SPHERICAL TRIGONOMETRY

For

Honours and Post-graduate Students

OF

All Indian Universities and for various competitive examinations.

By

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AUTHOR OF

A series of eleven books for Post-graduate classes ; A series of eleven books for degree classes; A series of seven books for Intermediate and Higher Secondary Examinations in Hindi & English ; and A series of four books for Roorkee and Kharagpur Entrance Examinations.

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PREFACE TO THE FIRST EDITION

It gives me great pleasure in bringing out my book on Spherical Trigonometry which I had announced much earlier. Owing to heavy rush of work I could not see it through the Press and I had to disappoint so many students who made innumerable enquiries about the same.

Spherical Trigonometry is the most important subject for the students of Astronomy and I am sure that as usual the present book is upto their expectation. The side *AB* of a spherical triangle is denoted by small *c* whereas the angle opposite to it is denoted by *C*. In spite of the best efforts put in by the compositors of M/s Prakash Printing Press I am afraid there may be some misprints of the above type which the students shall be easily able to correct themselves.

The various fundamental formulae have been deduced in an easy-to-understand manner and they are illustrated by a large number of solved examples followed by some unsolved examples for the practice of the students.

Any suggestions for the improvement of the book from any quarter will be highly appreciated.

315, Chhipi Tank,
Meerut.

M. L. KHANNA

PREFACE TO THE SECOND EDITION

The book has been thoroughly revised with the result that a few new questions have been added at various places. References of University papers have been given along with the questions. Latest University papers have also been added in the end.

M. L. KHANNA

Where is What?

LISTS OF IMPORTANT FORMULAE

Definitions

1. Arc ab of a small circle

$$=\text{Parallel arc } AB \text{ of a great circle} . \cos \phi$$

or Arc AB of a great circle

$$=\text{Parallel arc } ab \text{ of a small circle} . \sec \phi,$$

where ϕ is the arc Aa. [Refer fig. 9 P. 8]

2. The sides and the angles of a polar triangle are respectively supplements of the angles and sides of the primitive triangle,

i.e. $$a'=\pi-A,\ b'=\pi-B,\ c'=\pi-C$$

and $$A'=\pi-a,\ B'=\pi-b,\ C'=\pi-c.$$

Fundamental Formulae

3. **Cosine formula.**

$$\cos a=\cos b \cos c+\sin b \sin c \cos A,$$
$$\cos b=\cos c \cos a+\sin c \sin a \cos B,$$
$$\cos c=\cos a \cos b+\sin a \sin b \cos C,$$

or

$$\cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c},$$

$$\cos B=\frac{\cos b-\cos c \cos a}{\sin c \sin a},$$

$$\cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b}.$$

4. **Supplemental cosine formula.**

$$\cos A=-\cos B \cos C+\sin B \sin C \cos a,$$
$$\cos B=-\cos C \cos A+\sin C \sin A \cos b,$$
$$\cos C=-\cos A \cos B+\sin A \sin B \cos c,$$

or

$$\cos a=\frac{\cos A+\cos B \cos C}{\sin B \sin C},$$

$$\cos b=\frac{\cos B+\cos C \cos A}{\sin C \sin A},$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B},$$

5. **Sine formulae.**

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \sin b \sin c}$$

where $4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c$

and $\quad n^2 = \sin s \sin (s-a) \sin (s-b) \sin (s-c)$.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{2N}{\sin A \sin B \sin C}$$

where $\quad N^2 = -\cos S \cos (S-A) \cos (S-B) \cos (S-C)$.

6. **Formulae for Half Angle.**

$$\sin \frac{A}{2} = \sqrt{\left(\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}\right)},$$

$$\cos \frac{A}{2} = \sqrt{\left(\frac{\sin s \sin (s-a)}{\sin b \sin c}\right)},$$

$$\tan \frac{A}{2} = \sqrt{\left(\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}\right)}.$$

7. **Formulae for half a side.**

$$\sin \frac{a}{2} = \sqrt{\left(-\frac{\cos S \cos (S-A)}{\sin B \sin C}\right)},$$

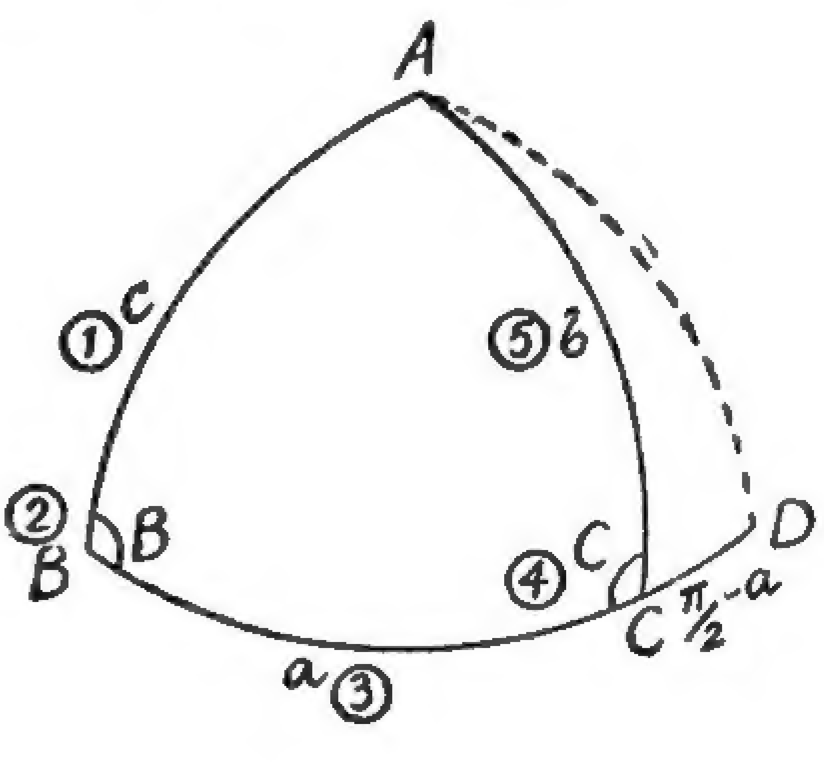
$$\cos \frac{a}{2} = \sqrt{\left(\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}\right)},$$

$$\tan \frac{a}{2} = \sqrt{\left(-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}\right)}.$$

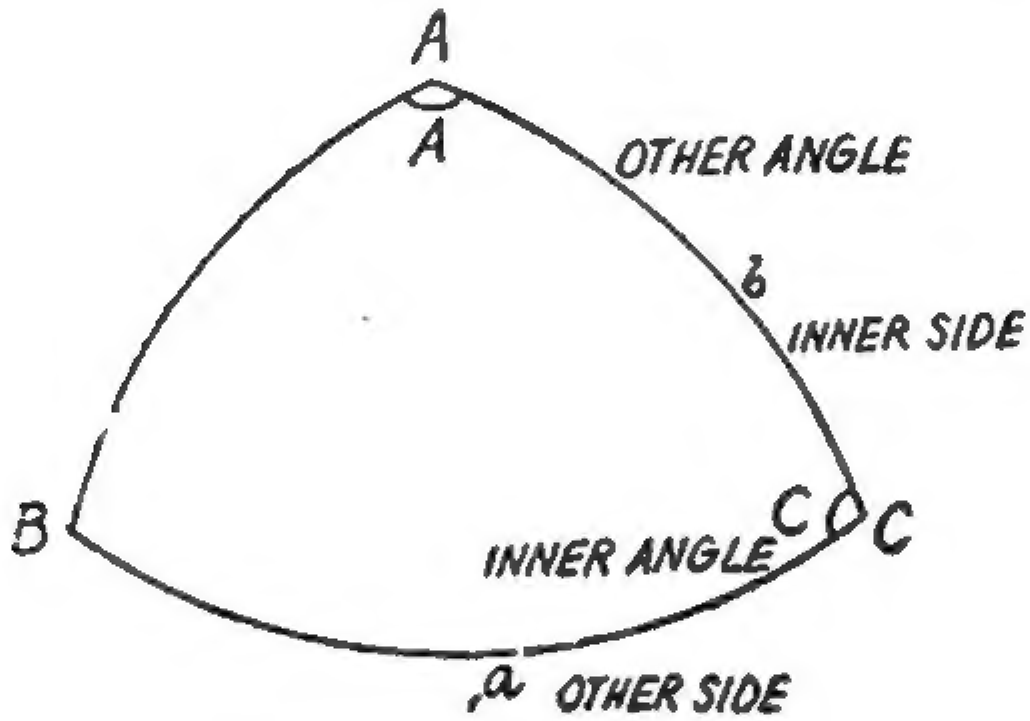
8. **Sine-cosine formula.**

$$\underset{1}{\sin c} \underset{2}{\cos B} = \underset{3}{\sin a} \underset{5}{\cos B}$$

$$- \underset{3}{\cos a} \underset{5}{\sin b} \underset{4}{\cos C}$$



9. The cotangent formula. Consecutive four.



cos (inner side) cos (inner angle)

=sin (inner side) cot (other side)

−sin (inner angle) cot (other angle)

$$\cos b \cos C = \sin b \cos a - \sin C \cot A.$$

10. Napier's Analogies.

(1) $$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2}$$

(2) $$\tan\frac{A-B}{2} = \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2}$$

(3) $$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}$$

(4) $$\tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}.$$

11. De Alembert's Analogies :

1. $$\frac{\sin \frac{A+B}{2}}{\cos \frac{C}{2}}=\frac{\cos \frac{a-b}{2}}{\cos \frac{c}{2}}.$$

2. $$\frac{\sin \frac{A-B}{2}}{\cos \frac{C}{2}}=\frac{\sin \frac{a-b}{2}}{\sin \frac{c}{2}}.$$

3. $$\frac{\cos \frac{A+B}{2}}{\sin \frac{C}{2}}=\frac{\cos \frac{a+b}{2}}{\cos \frac{c}{2}}.$$

4. $$\frac{\cos \frac{A-B}{2}}{\sin \frac{C}{2}}=\frac{\sin \frac{a+b}{2}}{\sin \frac{c}{2}}.$$

Right-Angled Triangle

12. Napier's Rule for Right-Angled Triangle.

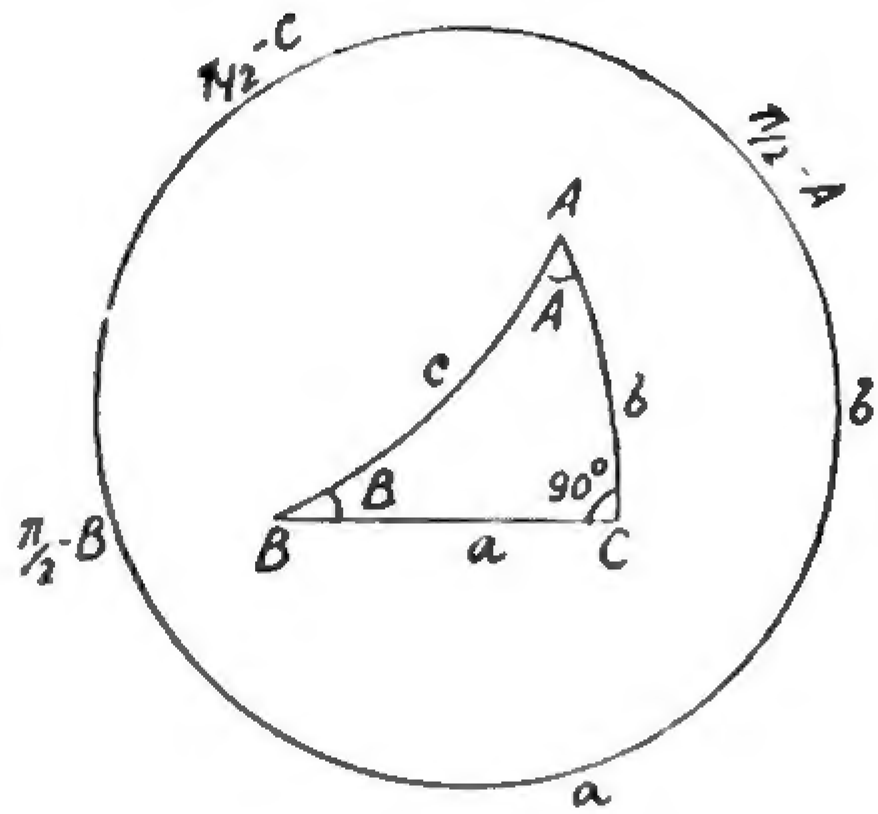
Sin of middle part

=Product of tangents of adjacent parts

=Product of cosines of opposite parts.

The sides containing the right angle are to be taken as they are, but for the remaining three elements we have to take their complements. If we choose any element as middle, then the two elements just adjacent to it on either side of it will be called adjacents whereas the remaining

two will be called opposites, *e g.*, if we choose c as middle then A and B will be adjacents and a, b will be opposites. While applying Napier's rules c will be taken $\pi/2-c$ and so also A and B will be taken as $\pi/2-A$ and $\pi/2-B$, whereas a and b containing the right angle C will be taken as they are.



$$\sin(\pi/2-c)=\tan(\pi/2-A)\tan(\pi/2-B)$$
$$=\cos a\cos b$$

or
$$\cos c=\cot A\cot B$$

or
$$\mathbf{\cos c=\cos a\cos b.}\quad \textbf{(V. Imp.)}$$

Similarly choosing a as middle, we have, as explained above,

$$\sin a=\tan b\tan(\pi/2-B)=\cos(\pi/2-A)\cos(\pi/2-c)$$

or
$$\sin a=\tan b\cot B=\sin A\sin c.$$

Spherical Excess

13. $E=A+B+C-\pi=2S-\pi$ or $S=E/2+\pi/2.$

14. $$\sin\frac{E}{2}=\frac{\{\sin s\sin(s-a)\sin(s-b)\sin(s-c)\}^{1/2}}{2\cos a/2\cos b/2\cos c/2}.$$

15. $$\tan\frac{E}{4}=\left[\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}\right]^{1/2}.$$

16. $$\cos\frac{E}{2}=\frac{1+\cos a+\cos b+\cos c}{4\cos a/2\cos b/2\cos c/2}$$
$$=\frac{\cos^2 a/2+\cos^2 b/2+\cos^2 c/2-1}{2\cos a/2\cos b/2\cos c/2}.$$

Small Variations

17. $\Delta a = \cos C\, \Delta b + \cos B\, \Delta c + k \sin b \sin c\, \Delta A,$

$\Delta b = \cos A\, \Delta c + \cos C\, \Delta a + k \sin c \sin a\, \Delta B,$

$\Delta c = -\cos B\, \Delta a + \cos A\, \Delta b + k \sin a \sin b\, \Delta C,$

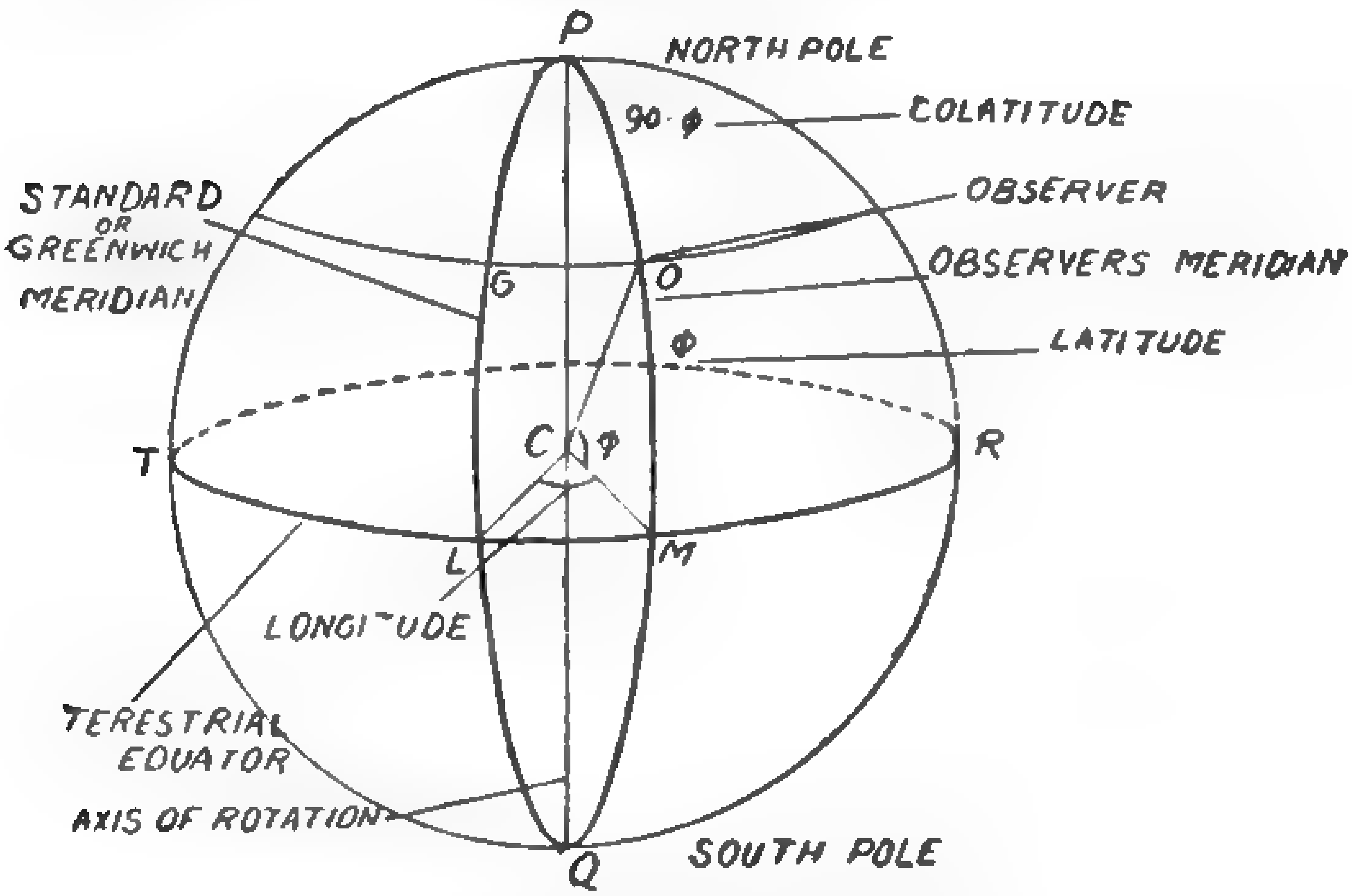
where $$k = \frac{\sin A}{\sin a}.$$

18. $\Delta A = -\cos c\, \Delta B - \cos b\, \Delta C + k^{-1} \sin B \sin C\, \Delta a,$

$\Delta B = -\cos a\, \Delta C - \cos c\, \Delta A + k^{-1} \sin C \sin A\, \Delta b,$

$\Delta C = \cos b\, \Delta A - \cos a\, \Delta B + k^{-1} \sin A \sin B\, \Delta c.$

19. Latitude and Longitude of a place on the earth.



In the above figure arc LM of the equator measured by $\angle LCM$ or spherical angle LPM is called longitude of O whereas arc $OM = \phi$ is called latitude of O. The arc $PO = 90 - \phi$ is colatitude of O.

CHAPTER I

DEFINITIONS

1. The section of a sphere by a plane is a circle.

Let O be the centre of the sphere whose radius is r. Let ABC be the section of the sphere by any plane. Through O draw OD perpendicular to the plane. Choose any two points A and B on the section and join OA, OB, DA and DB. Now OD is perpendicular to the plane and as such it is perpendicular to every line lying in the plane.

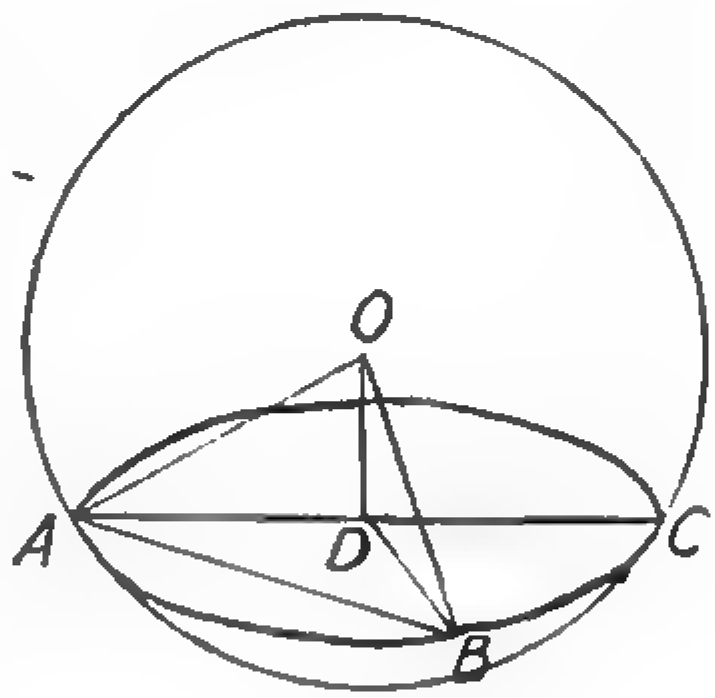


Fig. 1

$$\therefore \quad \angle ODA = \pi/2 = \angle ODB$$

or $\qquad AD^2 = OA^2 - OD^2 \quad$ or $\quad BD^2 = OB^2 - OD^2$.

But $OA = OB =$ radius of the sphere. $\therefore$ $AD = BD$ *i. e.* the point D is equidistant from any two points on the section which therefore is a circle whose centre D is the foot of the perpendicular from the centre of the sphere to the plane and whose radius is given by

$$AD = \sqrt{(OA^2 - OD^2)}$$

i. e. {(radius of sphere)^2 − (perp. from centre of sphere to the plane)^2}$^{1/2}$.

Note. *It is interesting to note here that if the plane passes through the centre of the sphere, then clearly $OD = 0$ and radius of the circle is equal to the radius of the sphere i. e. $AD = OA$ when $OD = 0$.*

2. Great circle and small circle.

We have proved above that the section of a sphere by a plane is a circle. If the plane cutting the sphere **passes through centre of the sphere** then the corresponding section is called a **great circle.** We have already proved above that the radius of a great circle is the same as that of the given sphere. Any other plane which does not pass through the centre of a sphere will cut the sphere in a circle which is called a small circle whose centre is the foot of the perpendicular from the centre of the given sphere to the plane which cuts the sphere and whose radius is determined as explained above. In the figure great circles and small circles are marked by G. C. and S C. respectively.

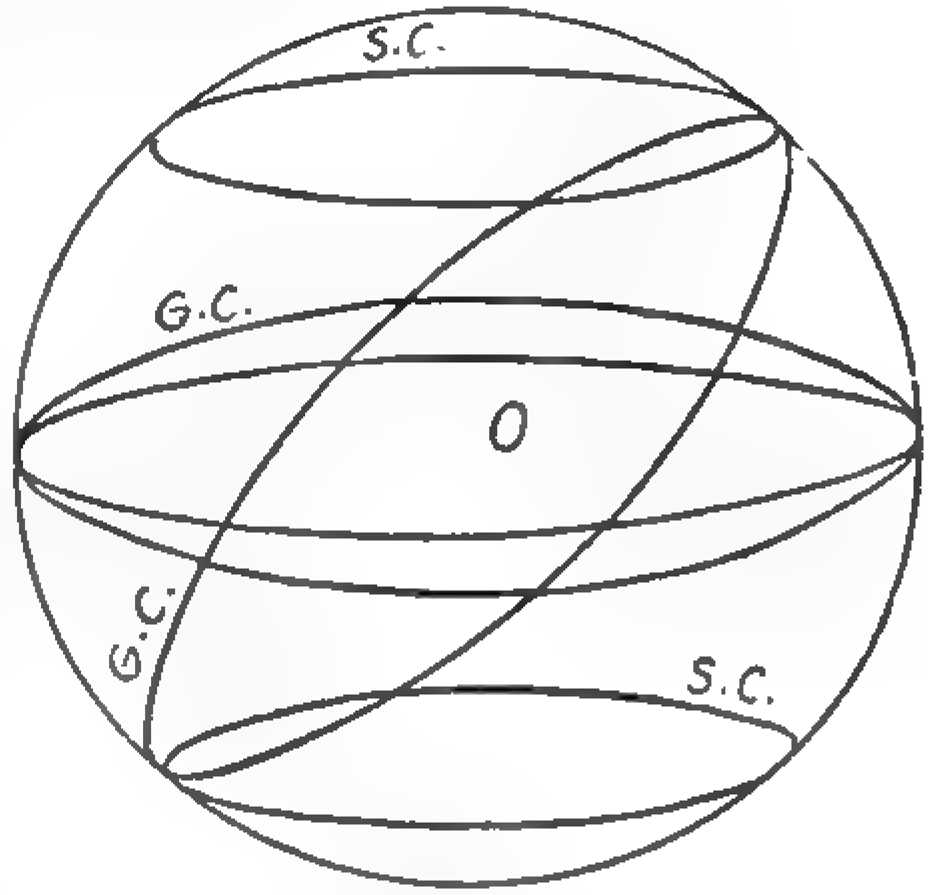


Fig. 2.

3. Number of great circles through two given points.

We know that through any three points not lying in the same straight line one and only one plane can be drawn and when the three points lie in the same straight line, then an infinite number of planes can be drawn. Hence if any two given points A and B on the surface of a sphere are not the extremities of a diameter *i. e.* A, B and O are not in the same straight line, then one and only one great circle can be drawn. If however A, B are the extremities of a diameter of the sphere *i. e.* A, B and O are in the same straight line,

then an infinite number of great circles can be drawn as shown below.

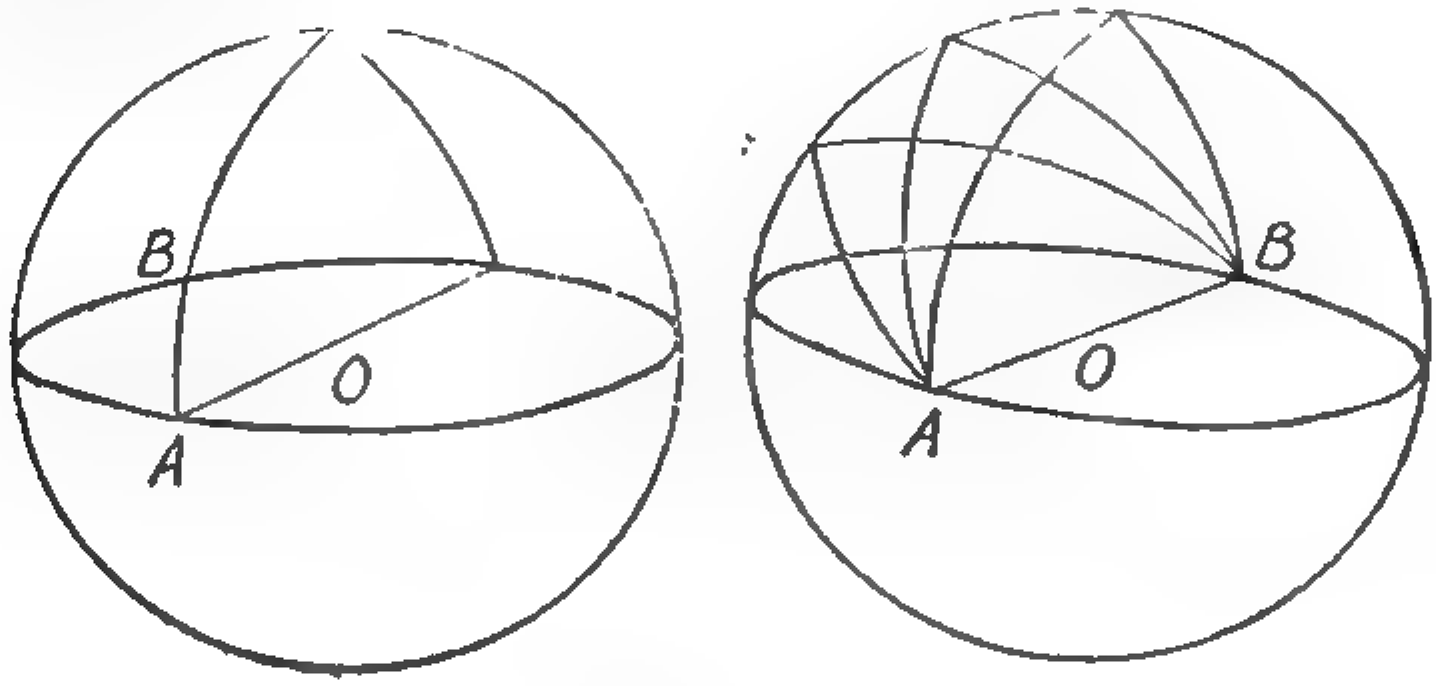


Fig. 3

4. Shortest arc joining two points on the surface of a sphere.

The shortest arc joining two points on the surface of a sphere should have the least curvature and as such its radius should be maximum *i.e.* equal to that of the sphere and hence it is the arc of a great circle through these two points. Whenever we speak of the arc of a great circle through any two points we shall always take the smaller of the two arcs in which the great circle is divided by those two points. Thus in the figure above by the arc of the great circle through *A* and *B* we mean the smaller thik-lined arc *AB*.

5. Two great circles bisect one another at their points of intersection.

Since the plane of each great circle passes through the centre hence their line of intersection should be a diameter of the sphere and therefore a diameter of the two great circles. Hence they bisect one another.

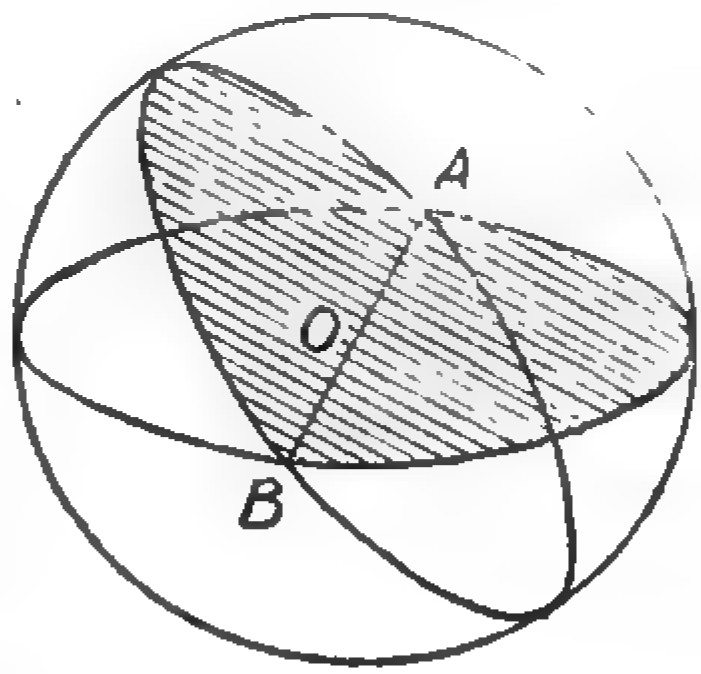


Fig. 4

6. The Axis and Pole.

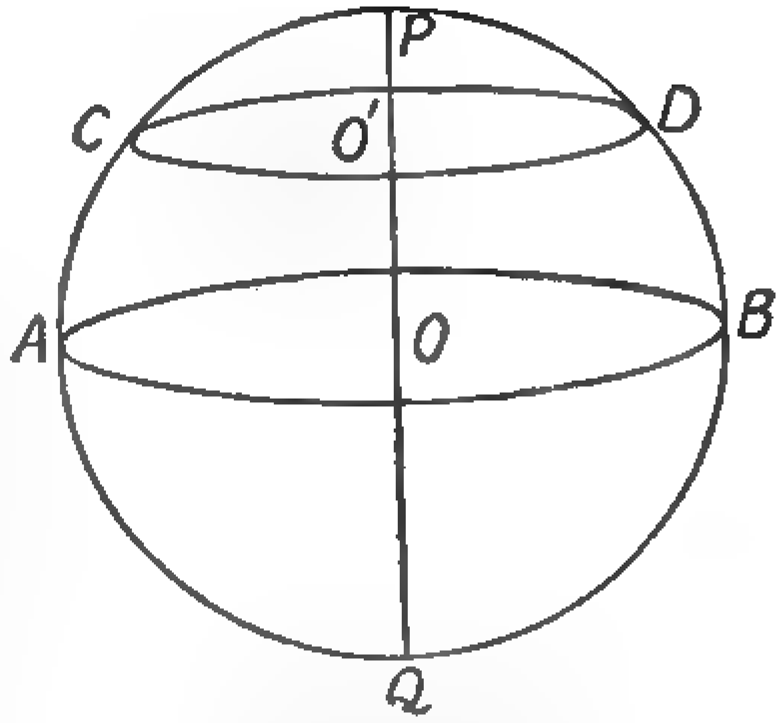


Fig. 5

The axis of any circle, great or small, is that diameter of the sphere which is perpendicular to its plane *i. e.* PQ is the axis in the adjoining figure. The extremities of this diameter are called the poles of the circles whose planes are perpendicular to this axis. It is easy to observe that in the case of a great circle the two poles are equidistant from the plane of the circle whereas in the case of a small circle one of the poles is nearer than the other from the plane of the circle. These poles are termed as nearer pole (or simply pole) and further pole (or nole) respectively.

7. Properties of poles.

The distance of a pole from every point on the circumference of the circle is same.

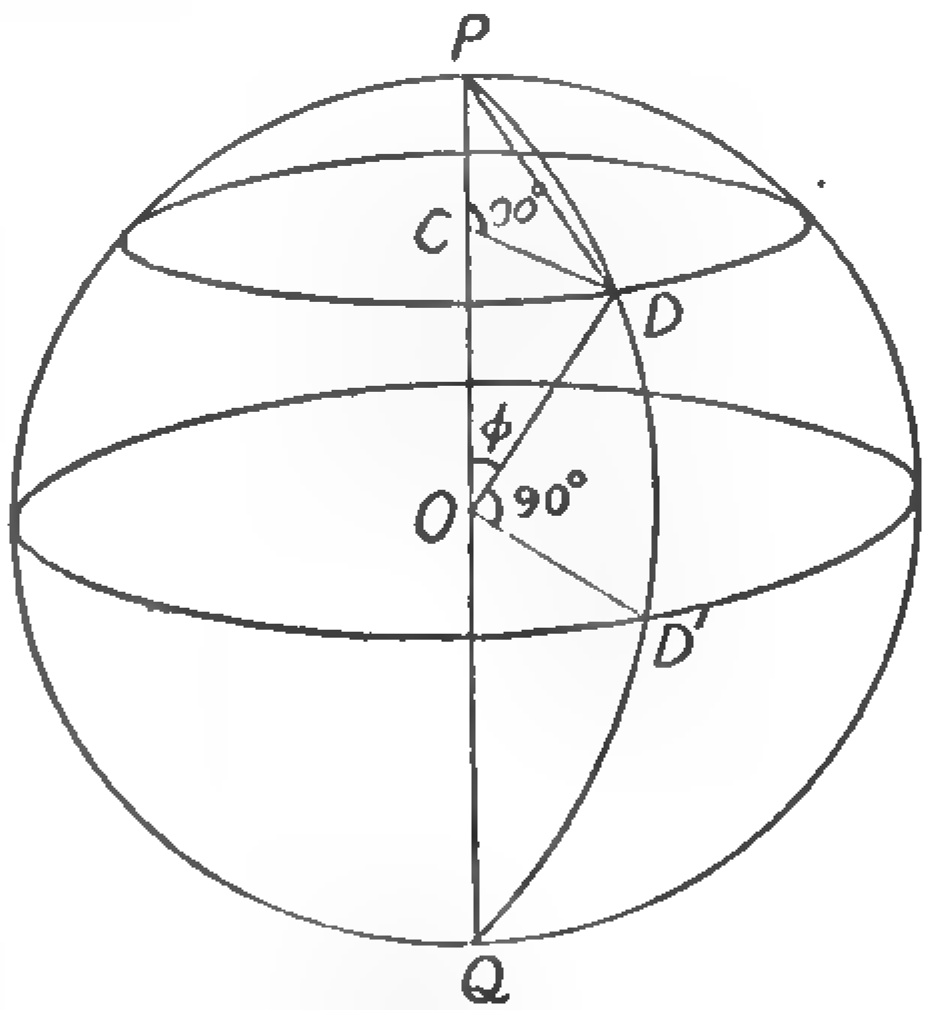


Fig. 6

Let P be the pole of a small circle whose centre is C, and D be any point on it. Join CD. Since PC is perpendicular to the plane of the circle, it is perpendicular to every line lying in the plane, *i e.* PC is perpendicular to CD. $\therefore \quad PD^2 = PC^2 + CD^2$. Now PC is the perpendicar distance of P from a given plane and CD being the

radius of small circle both are constant ; therefore PD is also constant. Hence the distance of P from any point on the circumference of a circle is constant.

Now draw a great circle through P and D ; then since chord PD is constant, therefore arc PD of the great circle is also constant and equal to angle ϕ subtended by the arc of the great circle through P and D at the centre of the sphere.

It is easy to note from here that the distance of the pole from any point on the circumference of a great circle is a quadrant *e. g.* if D' be any point on the circumference of a great circle whose pole is P, then clearly $\angle POD'=\pi/2$ or arc $PD'=\pi/2$.

8. Spherical radius.

Sherical radius of any small circle is the arc of the great circle intercepted between the nearer pole and any point on the small circle. Thus in the fig. the arc PD of the great circle is called the spherical radius of the small circle. Clearly spherical radius of the great circle is PD' which is a quadrant.

9. Angular distance.

The distance between any two points on a sphere is measured by the smaller arc of the great circle drawn through the two points. This arc is measured by the angle it subtends at the centre of the sphere.

10. Secondaries.

All those great circles which pass through the poles of a given circle are called secondaries to the given circles and clearly secondaries lie in a plane perpendicular to the plane of the given circles.

11. Spherical angle.

A spherical angle is the inclination of two arcs of great circles at their points of intersection on the surface of a sphere.

12. Measurement of a spherical angle.

(a) The spherical angle is the angle between the tangents to the arcs at their point of intersection.

(b) The spherical angle is the angle between the planes of the great circles which form the spherical angle.

(c) *The angle between two great circles (spherical angle) is measured by the arc intercepted by them on the great circle to which they are secondaries.*

Refer fig. 7. *PAQ*, *PDQ* and *PEQ* are secondaries to the great circle *AB*. The angle between *PAQ* and *PDQ* *i. e.* the angle *APD* is equal to the angle between their planes. *i.e.* the angle between two lines drawn in either plane perpendicular to the common line of intersection of the two planes. Clearly *OP* is perpendicular to both *OA* and *OD* both of which are in the planes of the two secondaries and hence $\angle AOD$ is measured by the arc *AD* and is equal to the spherical angle *APD*.

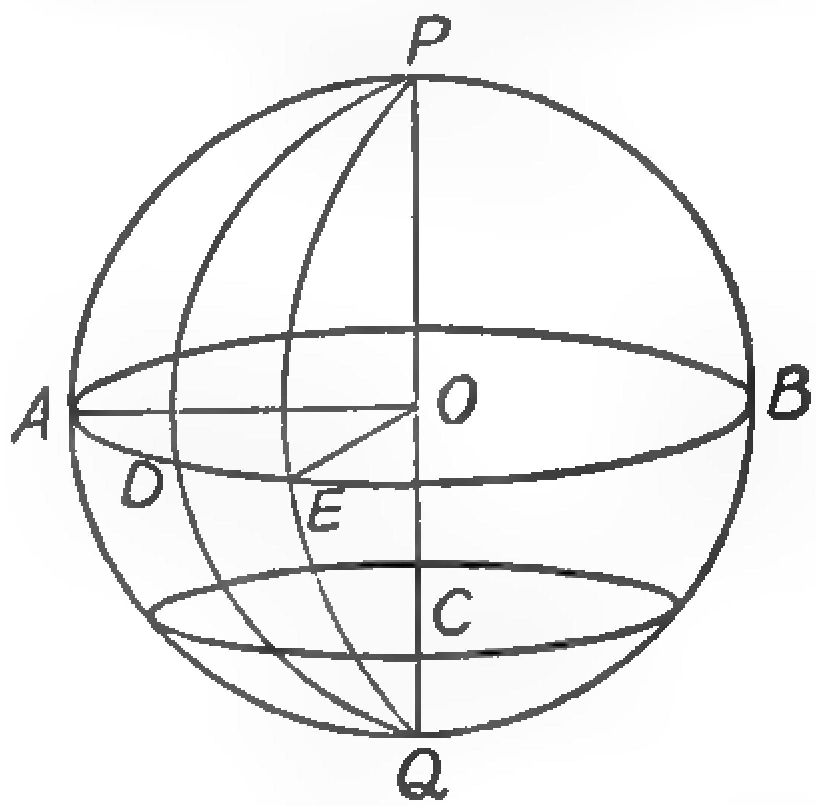


Fig. 7.

13. *If the arc of the great circles joining a point P on the surface of a sphere with two other points A and D on the surface of a sphere which are not at opposite extremities of a diameter, be each of them a quadrant, then P is the pole of the great circle through A and D.*

Refer fig. 7. Since *PA* and *PD* are both quadrants hence $\angle POA$ and $\angle POD$ are right angles *i e.* *PO* is perpendicular to both *OA* and *OD* *i. e.* *PO* is perpendicular to the plane *AOD*. Therefore *P* is the pole of the great circle through *A* and *D*.

14. *If from a point on the surface of a sphere there can be*

drawn two arcs of great circles not parts of the same great circle, the planes of which are at right angles to the plane of a given circle, that point is the pole of the given circle.

Refer fig. 7. O is the centre of the sphere and P any point on it. Let PD and PE be the two arcs of great circles drawn through P whose planes are perpendicular to the great circle AB. Clearly the line of intersection of their planes *i. e.* line PQ will be perpendicular to the plane of AB and as such it is the axis of the great circle AB. Therefore P is its pole.

In a similar manner P is the pole of any small circle whose plane is parallel to that of AB, *i. e.* perpendicular to the planes of PD and PE.

15. *The angle subtended at the centre of the sphere by the arc of a great circle which joins the poles of two great circles is equal to the inclination of the planes of the great circles. Hence this angle is equal to the angle between the great circles. In other words, we can say that the angular distance between the poles of two great circles is equal to their angle of intersection.*

Proof. Let CD and CE be the arcs of two great circles intersecting at C and the poles of the great circles through them be A and B respectively. Now draw a great circle through the poles A and B meeting the great circles in M and N respectively.

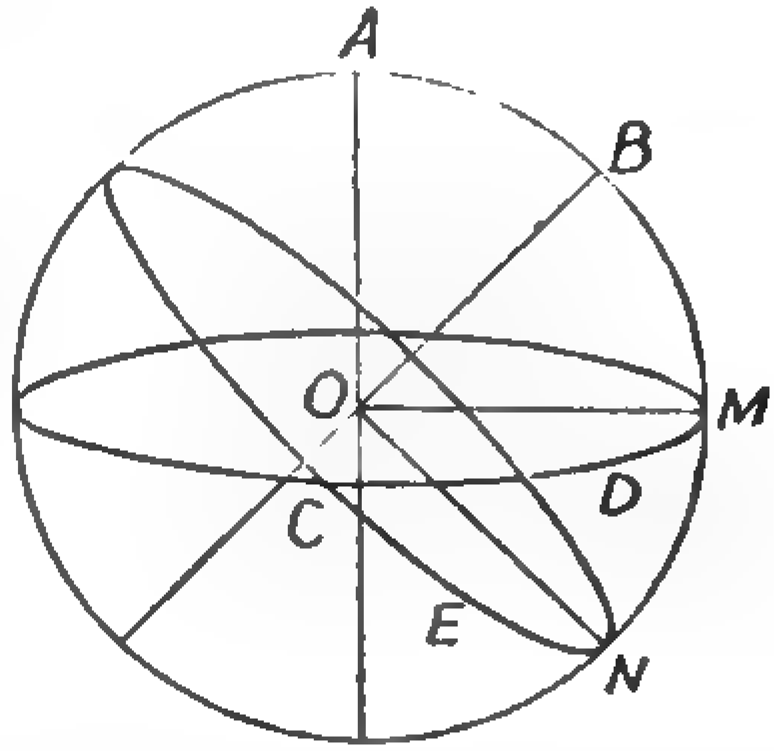


Fig. 8.

Since A is the pole of the great circle CDM, $\therefore$ AC is a quadrant. Similarly B is the pole of the great circle CEN, $\therefore$ BC is a quadrant. Thus both CA and CB are quadrants.

Hence by 14, C is the pole of the great circle through A and B, and as such CM and CN are both quadrants. Hence the great circles CDM and CEN are secondaries to the great circle AB and therefore the angle between CDM and CEN is measured by the arc of the great circle intercepted by them on the great circle to which they are secondaries [12 (c)]. Therefore the arc MN measures the angle between the great circles. Again AM is a quadrant and BN is a quadrant.

$$\therefore \quad AB = AM - BM = BN - BM = MN$$

Since arc $AB =$ arc MN, they will subtend equal angles at the centre O $\therefore$ $\angle AOB = \angle MON$.

Note :—From above we also conclude that the points of intersection of two great circles are the poles of the great circle passing through the poles of the given circles, just as C is the pole of the great circle (proved above) passing through A and B which are the poles of the two great circles intersecting at C. Similarly their other point of intersection is also the other pole.

16. Length of arc of a small circle.

[Agra 61, Nagpur 57, 61]

Let P be the pole of great and small circles with centres at O and C respectively. Let ab be the arc of a small circle. Through P and a and through P and b draw great circles meeting the great circle with centre O in A and B respectively. Clearly OP is perpendicular to Ca, Cb. OA and OB as the planes aCb and AOB are perpendicular to OP. Therefore Ca is parallel to OA and cb parallel to OB,

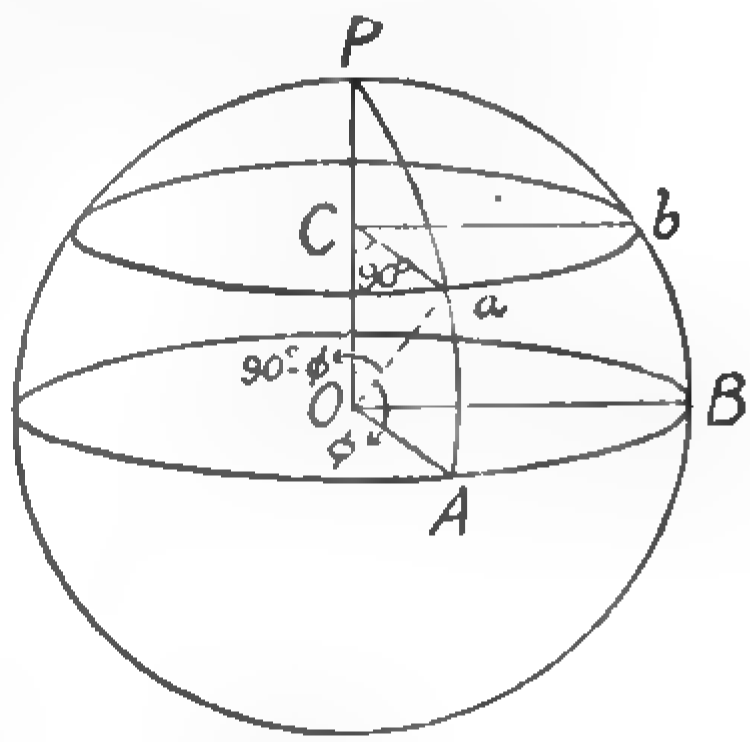


Fig. 9.

or $$\angle aCb = \angle AOB.$$

$$\because \quad \frac{\text{arc } ab}{\text{radius } Ca}=\frac{\text{arc } AB}{\text{radius } OA}$$

or $$\frac{\text{arc } ab}{\text{arc } AB}=\frac{\text{radius } Ca}{\text{radius } OA}=\frac{\text{radius } Ca}{\text{radius } Oa}, \quad \because \quad OA=Oa$$

or $$\frac{\text{arc } ab}{\text{arc } AB}=\sin POa=\cos AOa=\cos Aa=\cos \phi,$$

$$\textbf{arc ab}=\textbf{arc AB}\cos\phi$$

or **arc AB = arc ab sec ϕ where ϕ is the arc Aa.**

17. Spherical triangles.

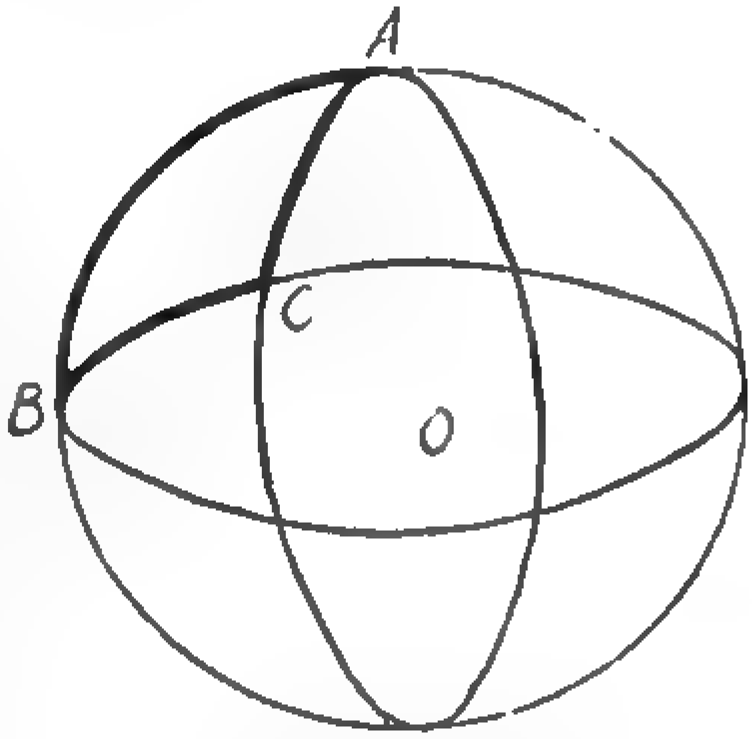


Fig. 10

Let A, B and C be any three points on the surface of a sphere. Join AB, BC and CA by arcs of the great circles passing through them. The figure thus formed is called spherical triangle ABC. We have already stated that by the arc of a great circle we mean the lesser arc which is intercepted between the point through which the circle passes. Therefore only one spherical triangle can be formed by joining the points A, B and C. Also the sides as in plane triangle, are denoted by a, b and c and are measured by the angles they subtend at the centre of the sphere. It follows therefore that the sides of a spherical triangle are each less than two right angles as none of them is greater than a semi-circle. The angle A *i.e.* $\angle BAC$ is the angle between the tangents to the great circles AB and AC drawn from A towards B and C respectively.

We shall show below that the angles of a spherical triangle cannot be greater than two right angles.

Let the triangle ABC be formed by arcs AB, BC and $CEDA$ having angle ABC greater than two right angles. Produce CB to meet the great circle at D. We know that two great circles bisect each other; therefore arc CED is a semicircle and as such the arc $CEDA$ of the spherical triangle is greater than two right angles which is contrary to what we have stated above that the sides of a spherical triangle are each less than two right angles. Hence none of the angles of a spherical triangle can be greater than two right angles.

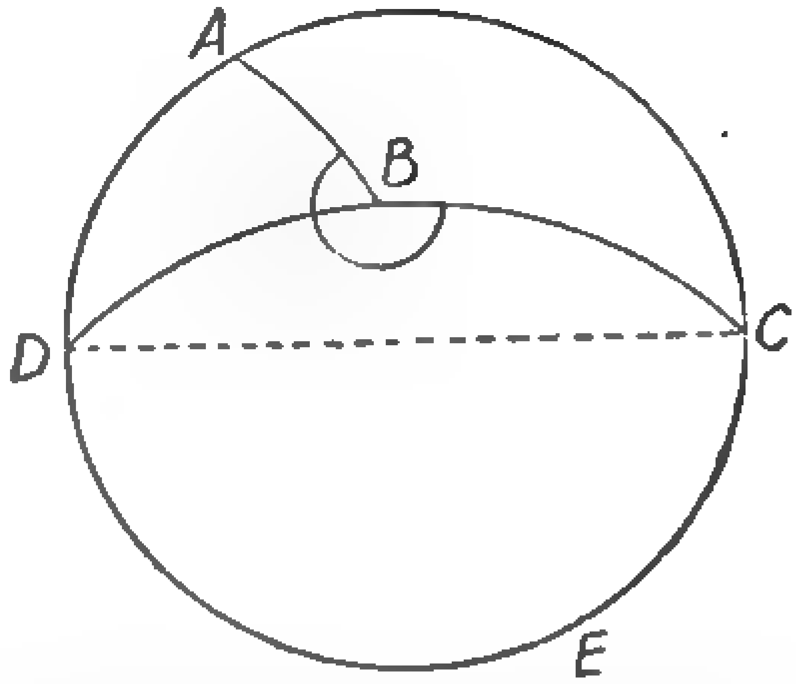


Fig. 11

18. Polar triangles.

Let ABC be a given spherical triangle and A', B', C' be the poles of the sides BC, CA, and AB respectively. The triangle formed by joining A', B', C' by great circle arcs is called the polar triangle of the given triangle which itself is called primitive triangle. Now we know that there are two poles to each circle and as such we can have eight polar triangles. But it is conventional to take that pole of a particular side which lies on the same side as the opposite angle. In this way we shall have only one polar triangle. Thus in the figure above, A', B', C' is the polar triangle of the triangle ABC.

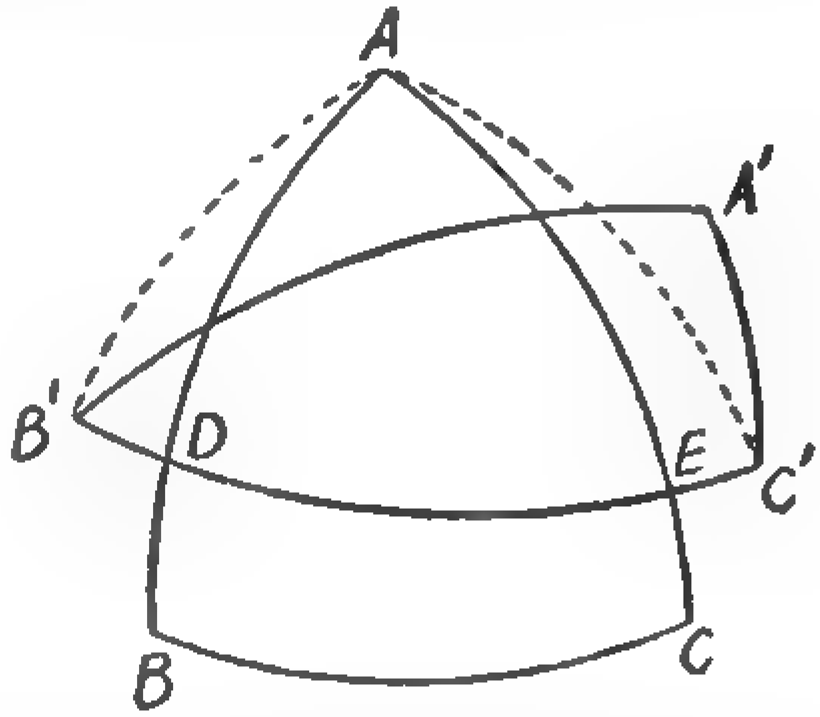


Fig. 12

19. Reciprocity of polarity.

If one triangle be the polar triangle of another triangle, then the latter will be the polar triangle of the former.

Refer fig. 12. B' is the pole of AC ; $\therefore$ AB' is a quadrant. Similarly C' is the pole of AB; $\therefore$ AC' is a quadrant *i.e.* both AB' and AC' are quadrants and hence by result 14, A is the pole of $B'C'$. Similarly B and C are the poles $C'A'$ and $A'B'$ **Hence proved.**

20. Relations between the sides and angles of polar triangles.

The sides and the angles of a polar triangle are respectively supplements of the angles and sides of the primitive triangle.

Refer fig. 12. Let ABC be a spherical triangle and $A'B'C'$ be the corresponding polar triangle. Again let AB and AC meet $B'C'$ (produced if necessary) in D and E respectively. Now we know that angle between two great circles is measured by the arc intercepted by them on a great circle to which they are secondaries [§ 12 (c)] Here A is the pole of $B'C'$; therefore the angle between AB and AC *i.e.* $\angle A=$ arc DE which these circles intercept on the circle $B'C'$ to which they are secondaries.

$$\therefore \quad \text{Arc } DE=\angle A. \qquad \ldots(1)$$

Again B' is the pole of CA ; $\therefore$ $B'E=\pi/2$;
C' is the pole of AB ; $\therefore$ $C'D=\pi/2$;

$$\therefore \quad B'E+C'D=\pi$$

or $\quad B'E+C'E+DE=\pi$ or $B'C'+DE=\pi$

or $\quad B'C'+\angle A=\pi$ or $B'C'=\pi-A$ [by (1)]

or $\quad a'=\pi-A$, and similarly $b'=\pi-B$ and $c'=\pi-C$.

Again ABC is the polar triangle of $A'B'C'$;

$$\therefore \quad a=\pi-A',\ b=\pi-B' \text{ and } c=\pi-C' ;$$

$$\therefore \quad A'=\pi-a,\ B'=\pi-b \text{ and } C'=\pi-c.$$

Above relation shows that in the case of polar triangles sides are supplements of the angles whereas the angles are the supplements of the sides of the given triangle.

21. Duality of theorem relating to spherical triangle.

If any relation holds good between the sides and angles of a spherical triangle, then the relation obtained by changing the sides into supplements of the angles and angles into supplements of the sides will also hold good.

Any relation between the sides and angles a, b, c, A, B, C of a given triangle ABC will also be true between the sides and angles a', b', c', A', B', C' of the polar triangle $A'B'C'$. Now we know that $A'=\pi-a$ and $a'=\pi-A$ etc. Hence the new relation obtained will be true.

22. Some properties of spherical triangles.

(a) *Any two sides of a spherical triangle are together less than the third side.*

By the arc of a great circle we mean the shorter arc of the great circle into which it is divided by the two points which are the extremities of the arc. Thus the arc AB is the shortest distance between the points A and B and is therefore less than $BC+CA$.

(b) *The sum of the three sides of a spherical triangle is less than the circumference of a great circle i e. 360°.*

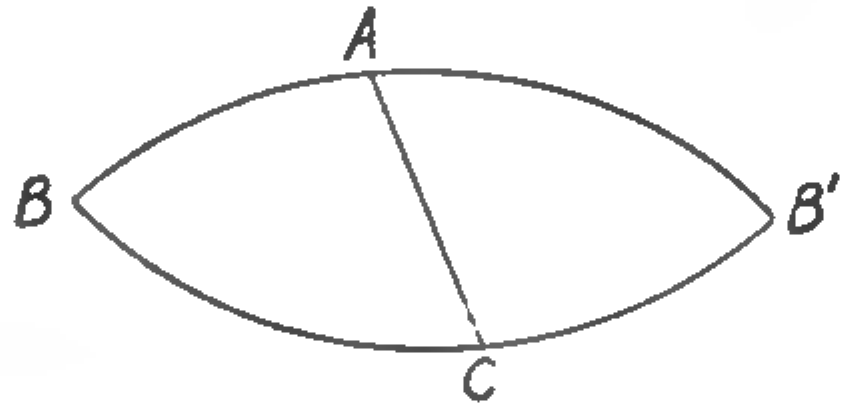


Fig. 13

Produce the sides BA and BC to meet at B'.

$\therefore \quad BAB' = BCB' = 180°$

as two great circles bisect each other. Now from $\triangle Ab'C$ we know by (a) that

$$AB'+CB' > AC.$$

Add $BA+BC$ to both sides.

$$(BA+AB')+(BC+CB') > BA+BC+AC$$

or $$BAB'+BCB' > AB+BC+CA$$

or $$180+180 > a+b+c \quad \text{or} \quad a+b+c < 360°.$$

(c) *The sum of the three angles of a spherical triangle is greater than two right angles but less than six right angles.*

We know that each of the angles of a spherical triangle is less than π; hence $A+B+C < 3\pi$ *i. e.* $<$ six right angles.

Again if a', b', c' be the sides of a polar triangle, then

$$a'+b'+c' < 2\pi \quad \text{[by (b)]}$$

or $\quad \pi-A+\pi-B+\pi-C < 2\pi \quad$ or $\quad \pi < A+B+C$

or $\quad A+B+C > \pi \quad$ *i. e.* two right angles.

(d) The angles at the base of an isosceles triangle are equal.

(e) The sides opposite to equal angles of a spherical triangle are equal.

(f) The side opposite to the greater angle of a spherical triangle is greater than the side opposite to the smaller angle.

(g) The angle opposite to the greater side of a spherical triangle is greater than the angle opposite to the smaller side.

(h) Two triangles drawn on a sphere are said to be congruent when they have similar curvatures, and also

(i) Two sides and the included angle of one are respectively equal to the two sides and the included angle of the other, or

(ii) If the three sides of one are respectively equal to the three sides of the other, or

(iii) If a side and the adjacent angles of one are respectively equal to the side and the adjacent angle of the other, or

(iv) When the three angles of one are respectively equal to the angles of the other.

CHAPTER II

FUNDAMENTAL FORMULÆ

Formulæ involving sides and angles of a spherical triangle.

§ 1. **The cosine formula.** *To find the value of cosine of an angle of a spherical triangle in terms of cosines and sines of the sides.* **(Bombay 61)**

Let AB, BC and CA be the arcs of a great circle whose centre is O, thus forming a spherical triangle ABC. We know that the sides of a spherical triangle are measured by the angles subtended by them at the centre.

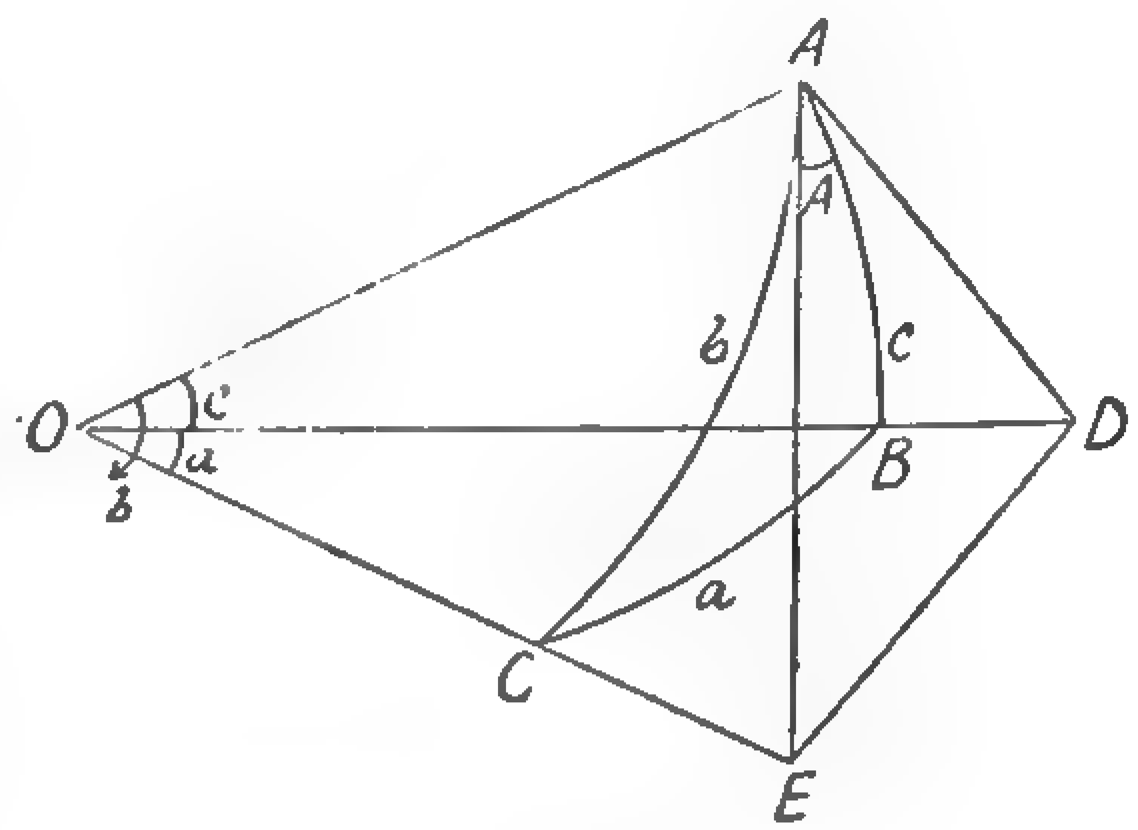


Fig. 14

$\therefore \quad \angle BOC = a$, $\angle AOB = c$ and $COA = b$.

Again we know that angle between two curves is the angle between the tangents to them at their common point of intersection. Hence if AE and AD be the tangents to the great circle arcs AC and AB containing the angle A and meeting OC and OB produced respectively in E and D, then $\angle DAE = \angle A$. We have here assumed that the sides containing the angle A are each less than $\pi/2$, for otherwise the construction will fail as the tangents at A will not meet OC and OB produced.

$\therefore \quad \angle EOD = \angle BOC = a, \quad \angle AOD = \angle AOB = c,$
$\angle AOE = \angle AOC = b.$

Now from plane triangles DEO and DEA we get by cosine formula,

$a^2 = b^2 + c^2 - 2bc \cos A.$

$DE^2 = OD^2 + OE^2 - 2OD.OE \cos a$ [from $\triangle$ DEO]...(1)

and $DE^2 = AD^2 + AE^2 - 2AD.AE \cos A.$ [from $\triangle$ DEA]...(2)

Now we know that angle between a tangent and the radius is a right angle.

$\therefore \quad \angle OAD = \pi/2$ or $OD^2 = OA^2 + AD^2$ ...(3)

$\angle OAE = \pi/2$ or $OE^2 = OA^2 + AE^2$ ...(4)

Subtracting (1) and (2), we get

$$0 = (OD^2 - AD^2) + (OE^2 - AE^2) - 2OD\ OE.\cos a + 2AD.AE \cos A$$

or $$2OD.OE.\cos a = OA^2 + OA^2 + 2AD.AE \cos A$$
[by (3) and (4)].

Cancelling (2) and dividing by $OD.OE$, we get

$$\cos a = \frac{OA}{OD}\cdot\frac{OA}{OE} + \frac{AD}{OE}\cdot\frac{AE}{OE} \cos A \qquad ...(5)$$

From right-angled triangles OAD and OAE, we have

$$\frac{OA}{OD} = \cos c \text{ and } \frac{AD}{OD} = \sin c, \ \frac{OA}{OE} = \cos b \text{ and } \frac{AE}{OE} = \sin b \ ...(6)$$

Hence from (5) with the help of (6), we have

$$\mathbf{\cos a = \cos b \cos c + \sin b \sin c \cos A} \qquad ...(7)$$

Similarly we can prove that

$$\mathbf{\cos b = \cos c \cos a + \sin c \sin a \cos B} \qquad ...(8)$$

and $$\mathbf{\cos c = \cos a \cos b + \sin a \sin b \cos C} \qquad ...(9)$$

A careful look on the three formulae will help the students in easily remembering the above formulae which are very widely used in Spherical Trigonometry and Astronomy.

Another form.

The above three formulae can be written in another form as below :—

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \text{ [from (7)]} \quad \ldots(10)$$

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a} \text{ [from (8)]} \quad \ldots(11)$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} \text{ [from (9)]} \quad \ldots(12)$$

Note. We have while proving the above formula assumed that the sides containing the angle A are each less than $\pi/2$. Now we shall show that whatever the sides may be the above formula is true.

Case I. The side AB containing the angle is greater than $\pi/2$.

Produce BA and BC to meet at B' where $BB' = \pi$.

$\therefore$ $AB' = \pi - c$ which is less than $\pi/2$ as c is greater than $\pi/2$.

The side AC is already assumed to be less than $\pi/2$.

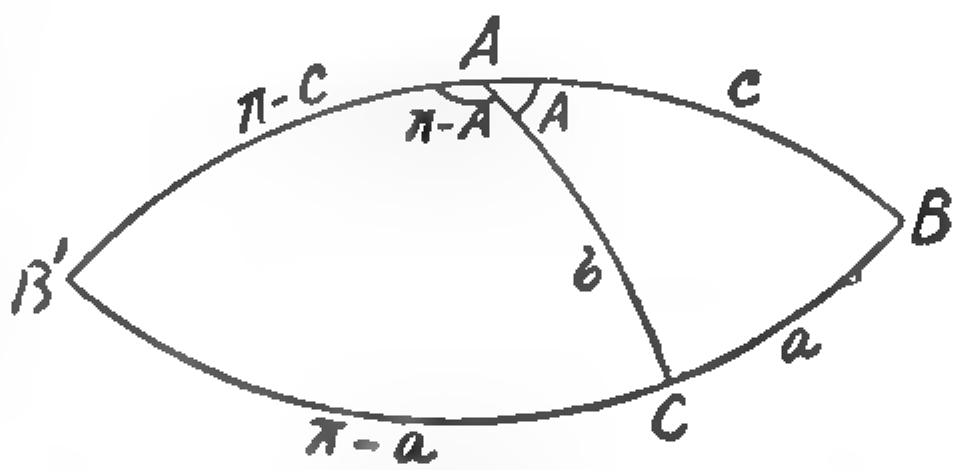


Fig 15

Also $CB' = \pi - a$ and $\angle B'AC = \pi - A$.

Now in spherical triangle $B'AC$ the sides containing the angle $B'AC$ *i.e.* $\pi - A$ are each less than $\pi/2$ and hence we have by cosine formula,

$$\cos(\pi - a) = \cos b \cos(\pi - c) + \sin b \sin(\pi - c) \cos(\pi - A)$$

or $\quad -\cos a = -\cos b \cos c - \sin b \sin c \cos A$

or $\quad \cos a = \cos b \cos c + \sin b \sin c \cos A$

which proves (7).

Case II. Both the sides AB and AC containing the angle A are greater than $\pi/2$.

In the case produce AB and AC to meet at A'.

$\therefore \quad \angle CA'B = A$ and $CA' = \pi - b$ which is less than $\pi/2$ as b is greater than $\pi/2$ and similarly $BA' = \pi - c$ is less than $\pi/2$.

Now in spherical triangle $BA'C$ the sides containing the angle A at A' are each less than $\pi/2$ and hence we have by cosine formula

$$\cos a = \cos(\pi - b)\cos(\pi - c) + \sin(\pi - b)\sin(\pi - c)\cos A$$

or $\cos a = \cos b \cos c + \sin b \sin c \cos A$ which proves (7).

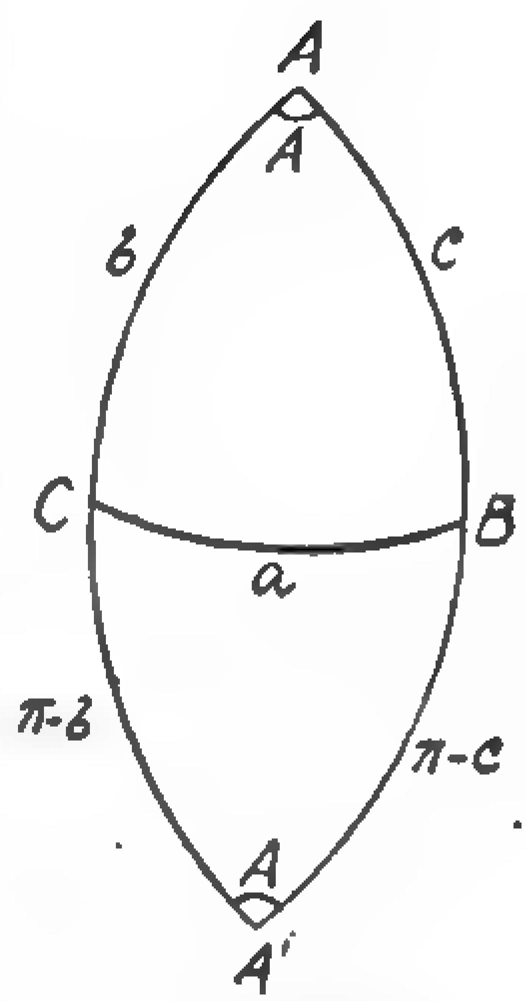


Fig. 16

§ 2. **Supplemental Cosine Formula;** *To find the value of cosine of the side of a spherical triangle in terms of cosines and sines of the angle.* **(Nagpur 56)**

Let there be a spherical triangle ABC and its polar triangle be $A'B'C'$; then we have already proved that the **sides** and the **angles** of a triangle are supplements of the angles and the sides of the corresponding polar triangle.

$$\therefore \quad A' = \pi - a, \quad B' = \pi - b, \quad C' = \pi - c$$

and

$$a' = \pi - A, \quad b' = \pi - B, \quad c' = \pi - C.$$

Now applying cosine formula on the polar triangle $A'B'C'$, we have

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A' \quad [\S\ 1\ \text{R. } 7]$$

or

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a)$$

$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a)$$

or

$$\mathbf{\cos A = -\cos B \cos C + \sin B \sin C \cos a.} \quad \ldots(1)$$

Similarly we can say that

$$\mathbf{\cos B = -\cos C \cos A + \sin C \sin A \cos b,} \quad \ldots(2)$$

$$\mathbf{\cos C = -\cos A \cos B + \sin A \sin B \cos c.} \quad \ldots(3)$$

The above three formulae are quite similar to formulae Nos. 7, 8 and 9 of § 1, except minus sign with the first term in R H S.

Another Form.

The above formulae can be written in another form as below :—

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \quad \text{from (1)} \quad \ldots(4)$$

$$\cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A} \quad \text{from (2)} \quad \ldots(5)$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B} \quad \text{from (3)} \quad \ldots(6)$$

They are quite similar to formulae Nos. 10, 11, 12 of § 1, except that here we have plus sign in the numerator and there we had minus sign.

§ 3. **Sine Formula.** *To prove that the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides, i. e.*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \qquad \textbf{(Utkal 57)}$$

Note the similarity of this formula with the corresponding sine formula of plane trigonometry which states that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Proof. We have proved in § 1 result 10 that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

$$\therefore \quad \sin^2 A = 1 - \cos^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}.$$

or $$\sin^2 A=\frac{(1-\cos^2 b)(1-\cos^2 c)-(\cos^2 a+\cos^2 b\cos^2 c-2\cos a\cos b\cos c)}{\sin^2 b\sin^2 c}$$

or $$\sin^2 A=\frac{(1-\cos^2 b-\cos^2 c+\cos^2 b\cos^2 c)-(\cos^2 a+\cos^2 b\cos^2 c-2\cos a\cos b\cos c)}{\sin^2 b\sin^2 c}$$

or $$\sin A=\frac{(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)^{1/2}}{\sin b\sin c}.$$

We have taken only +ive sign with the radical as we know that the angles and the sides of a spherical triangle are each less than two right angles and as such $\sin A$, $\sin b$ and $\sin c$ are all +ive.

$$\therefore \quad \frac{\sin A}{\sin a}=\frac{(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)^{1/2}}{\sin a\sin b\sin c}.$$

The symmetry of the result shows that

$$\frac{\sin \mathbf{A}}{\sin \mathbf{a}}=\frac{\sin \mathbf{B}}{\sin \mathbf{b}}=\frac{\sin \mathbf{C}}{\sin \mathbf{c}}$$

$$=\frac{(1-\cos^2 \mathbf{a}-\cos^2 \mathbf{b}-\cos^2 \mathbf{c}+2\cos \mathbf{a}\cos \mathbf{b}\cos \mathbf{c})^{1/2}}{\sin \mathbf{a}\sin \mathbf{b}\sin \mathbf{c}}.$$

Note. The expression

$$1-\cos^2 \mathbf{a}-\cos^2 \mathbf{b}-\cos^2 \mathbf{c}+2\cos \mathbf{a}\cos \mathbf{b}\cos \mathbf{c}=4\mathbf{n}^2.$$

$$\therefore \quad \frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}=\frac{2n}{\sin a\sin b\sin c}.$$

In determinant form,

$$4n^2=\begin{vmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{vmatrix}.$$

§ **4. Formula for half angle.** *To find the value of sine, cosine and tangent of half an angle of the spherical triangle in terms of functions of the sides.*

We have already proved in § 1 Result 10 that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Now we know that

$$2 \sin^2 A/2 = 1 - \cos A$$

or $$2 \sin^2 \frac{A}{2} = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

or $$2 \sin^2 \frac{A}{2} = \frac{(\cos b \cos c + \sin b \sin c) - \cos a}{\sin b \sin c}$$

or $$2 \sin^2 \frac{A}{2} = \frac{\cos (b-c) - \cos a}{\sin b \sin c}$$

or $$2 \sin^2 \frac{A}{2} = \frac{2 \sin \frac{b-c+a}{2} \sin \frac{a-b+c}{2}}{\sin b \sin c}.$$

If we take that $2s = a+b+c$

and then $a+b-c = a+b+c-2c = 2(s-c)$

and $a+c-b = a+b+c-2b = 2(s-b)$;

$$\therefore \quad \sin^2 \frac{A}{2} = \frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}$$

or $$\sin \frac{A}{2} = \sqrt{\left[\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}\right]}. \qquad \ldots(1)$$

Note. We have taken the +ive sign with the radical as we know that A is less than two right angles *i. e.* $A/2$ is less than a right angle and hence its sine, cosine and tangent are all +ive.

Again we know that

$$2 \cos^2 A/2 = 1 + \cos A$$

or $$2 \cos^2 \frac{A}{2} = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

or $$2 \cos^2 \frac{A}{2} = \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c}$$

or $$2\cos^2\frac{A}{2}=\frac{\cos a-\cos(b+c)}{\sin b\sin c}$$

or $$2\cos^2\frac{A}{2}=\frac{2\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}}{\sin b\sin c}$$

or $$\cos^2\frac{A}{2}=\frac{\sin s.\sin(s-a)}{\sin b\sin c}$$

or $$\cos\frac{A}{2}=\sqrt{\left[\frac{\sin s\ \sin(s-a)}{\sin b\sin c}\right]} \quad \ldots(2)$$

(Nagpur 58)

$$\therefore\quad \tan\frac{A}{2}=\frac{\sin A/2}{\cos A/2}\sqrt{\left[\frac{\sin(s-b)\sin(s-c)}{\sin s.\sin(s-a)}\right]} \ldots(3)$$

Also we know that

$\sin A=2\sin A/2\cos A/2$ and from (1) and (2), we get

$$\sin A=\frac{2}{\sin b\sin c}\sqrt{[\sin s\sin(s-a)\sin(s-b)\sin(s-c)]}$$

or $$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}$$

$$=\frac{2}{\sin a\sin b\sin c}\cdot\sqrt{[\sin s\sin(s-a)\sin(s-b)\sin(s-c)]}$$

$$=\frac{2n}{\sin a\sin b\sin c}$$

where $\quad n^2=\sin s\sin(s-a)\sin(s-b)\sin(s-c)$

and n is called the norm of the sides of a spherical triangle.

Note. Equating the values of n^2 from § 3 Note P. 19 above, we get

$4\sin s\sin(s-a)\sin(s-b)\sin(s-c)$

$$=1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c.$$

§ 5. **Formula for half a side.** *To find the value of sine, cosine and tangent of half a side of a spherical triangle in terms of the functions of its angles.*

(Agra 57)

We have already proved in § 2 result 4 that

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

Now we know that $2 \sin^2 a/2 = 1 - \cos a$

or $$2 \sin^2 \frac{a}{2} = 1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

or $$2 \sin^2 \frac{a}{2} = -\frac{(\cos B \cos C - \sin B \sin C) + \cos A}{\sin B \sin C}$$

or $$2 \sin^2 \frac{a}{2} = -\frac{\cos (B + C) + \cos A}{\sin B \sin C}$$

or $$2 \sin^2 \frac{a}{2} = -\frac{2 \cos \dfrac{A+B+C}{2} \cos \dfrac{B+C-A}{2}}{\sin B \sin C}.$$

Let $B + C + A = 2S$; $\therefore$ $B + C - A = 2(S - A)$.

$$\therefore \sin \frac{a}{2} = \sqrt{\left\{ -\frac{\cos S . \cos (S - A)}{\sin B \sin C} \right\}}. \quad \ldots(1)$$

Again $2 \cos^2 \dfrac{a}{2} = 1 + \cos a$

or $$2 \cos^2 \frac{a}{2} = 1 + \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

or $$2 \cos^2 \frac{a}{2} = \frac{\cos A + (\cos B \cos C + \sin B \sin C)}{\sin B \sin C}$$

or $$2 \cos^2 \frac{a}{2} = \frac{\cos A + \cos (B - C)}{\sin B \sin C}$$

or $$2 \cos^2 \frac{a}{2} = \frac{2 \cos \dfrac{A+B-C}{2} . \cos \dfrac{A-B+C}{2}}{\sin B \sin C}$$

or $$\cos \frac{a}{2} = \sqrt{\left\{ \frac{\cos (S - B) \cos (S - C)}{\sin B \sin C} \right\}} \quad \ldots(3)$$

$$\therefore \quad \tan \frac{a}{2} = \frac{\sin \frac{a}{2}}{\cos \frac{a}{2}} = \sqrt{\left(-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}\right)} \quad \ldots(3)$$

Note. Carefully compare the form for the value of $\sin \frac{A}{2}$ and $\cos \frac{a}{2}$ and also of $\cos \frac{A}{2}$ and $\sin \frac{a}{2}$.

Again $\sin a = 2 \sin \frac{a}{2} \cos \frac{a}{2}$ and from (1) and (2), we get

$$\sin a = \frac{2}{\sin B \sin C} \sqrt{[-\cos S \cos (S-A) \cos (S-B) \cos (S-C)]}.$$

(Agra 60)

$$\therefore \quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{2N}{\sin A \sin B \sin C}$$

where $\qquad N^2 = -\cos S \cos (S-A) \cos (S-B) \cos (S-C).$

Here N is called the norm of the angles of a spherical triangle.

§ 6. **Sine-cosine formula.** *Relation between three sides and two angles of a spherical triangle.*

(Vikram 59, Bombay 61)

We are going to establish the following relation between the five elements of the spherical triangle, namely, three sides and two angles. The way in which this formula is written should be clearly understood. The five elements have been numbered which will help the students to remember the formula ; while reading it you should look to the figure.

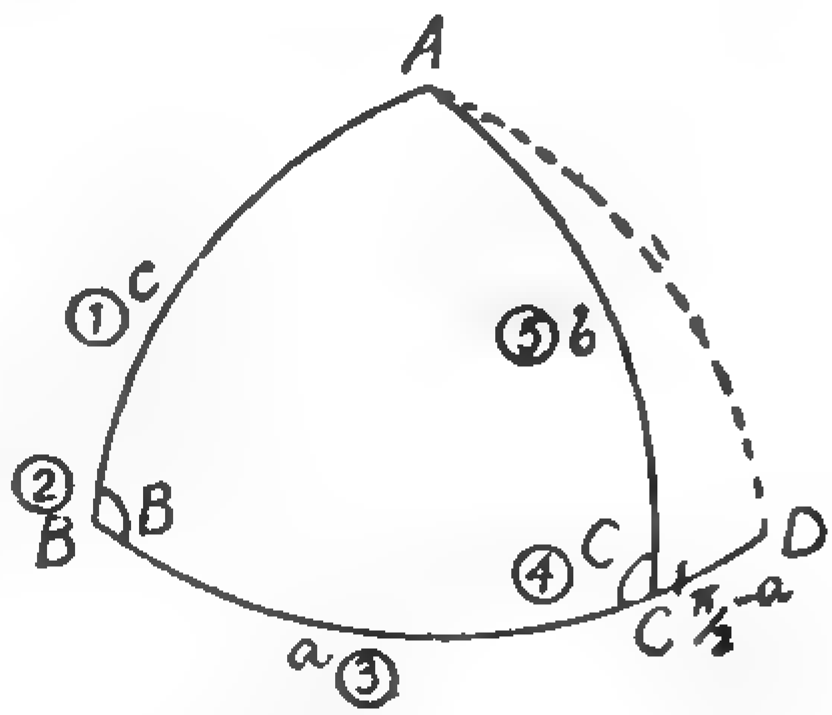


Fig. 17

$$\underset{1}{\sin c} \quad \underset{2}{\cos B} = \underset{3}{\sin a} \quad \underset{5}{\cos b} - \underset{3}{\cos a} \quad \underset{5}{\sin b} \quad \underset{4}{\cos C}$$

We can also say that

$$\sin c \ \cos A = \sin b \ \cos a - \cos b \ \sin a \ \cos C$$

or $$\underset{5}{\sin b} \ \underset{4}{\cos C} = \underset{3}{\sin a} \ \underset{1}{\cos c} - \underset{3}{\cos a} \ \underset{1}{\sin c} \ \underset{2}{\cos B}$$

or $$\sin b \ \cos A = \sin c \ \cos a - \cos c \ \sin a \ \cos B$$

Proof. We shall here prove the first formula, *i.e.*

sin c cos B=sin a cos b−cos a sin b cos C.

1st Method. Produce BC to D such that $BC=\pi/2$;

$$\therefore \quad CD=\pi/2-a.$$

Join AD. Now we shall equate the values of $\cos AD$ from the triangles ABD and ACD by using § 1, Result 7.

$$\cos AD=\cos c \cos 90+\sin c \sin 90 \cos B$$
$$=\sin c \cos B \text{ [from } \triangle\ ABD],$$
$$\cos AD=\cos b \cos(\pi/2-a)$$
$$+\sin b \sin(\pi/2-a)\cos(\pi-C) \text{ [from } \triangle\ ACD].$$
$$=\sin a \cos b-\cos a \sin b \cos C$$

Equating the values of $\cos AD$, we get

$$\sin c \cos B=\sin a \cos b-\cos a \sin b \cos C$$

which proves the result.

2nd Method.

$$\cos c=\cos a \cos b+\sin a \sin b \cos C, \qquad \ldots(1)$$

and $$\cos b=\cos c \cos a+\sin c \sin a \cos B. \qquad \ldots(2)$$

In the formula to be proved we do not require $\cos c$. Hence putting the value of $\cos c$ from (1) in (2),

$$\cos b=\cos a\,(\cos a \cos b+\sin a \sin b \cos C)+\sin c \sin a \cos B;$$

$$\therefore \quad \cos b\,(1-\cos^2 a)=\sin a \cos a \sin b \cos C$$
$$+\sin c \sin a \cos B.$$

Cancel $\sin a$ throughout.

$$\cos b \sin a=\cos a \sin b \cos C+\sin c \cos B;$$

$$\therefore \quad \sin c \cos B=\sin a \cos b-\cos a \sin b \cos C.$$

Hence proved.

§ 7. Supplemental sine cosine formula.

Let there be a spherical triangle ABC and its polar

triangle be A', B', C'; then as in § 2, $A'=\pi-a$ etc., and $a'=\pi-A$.

Now applying sine cosine formula to polar triangle $A'B'C'$, we have

$$\sin c' \cos B' = \sin a' \cos b' - \cos a' \sin b' \cos C'$$

or $$\sin(\pi-C)\cos(\pi-b)=\sin(\pi-A)\cos(\pi-B)$$
$$-\cos(\pi-A)\sin(\pi-B)\cos(\pi-c)$$

or $$-\sin C \cos b = -\sin A \cos B - \cos A \sin B \cos c$$

or $$\sin C \cos b = \sin A \cos B + \cos A \sin B \cos c.$$

The form of the above formula is same as in § 6 except that there is a +ive sign in R.H.S. whereas is § 6 there was −ive sign. The way to remember this is same as in § 6.

§ **8. The cotangent formula.** *Relation between two sides, included angle and another angle, i e. relation between consecutive four elements of a spherical triangle.*

[Utkal 56, 59, Bombay 61, Nagpur 57, 61]

Let us take the four consecutive elements a, C, b and A. The side b which comes between the two angles will be termed as inner side and the side a as other side. Similarly the angle C which comes between two sides will be termed as inner angle, and the angle A as other angle and we shall prove now that

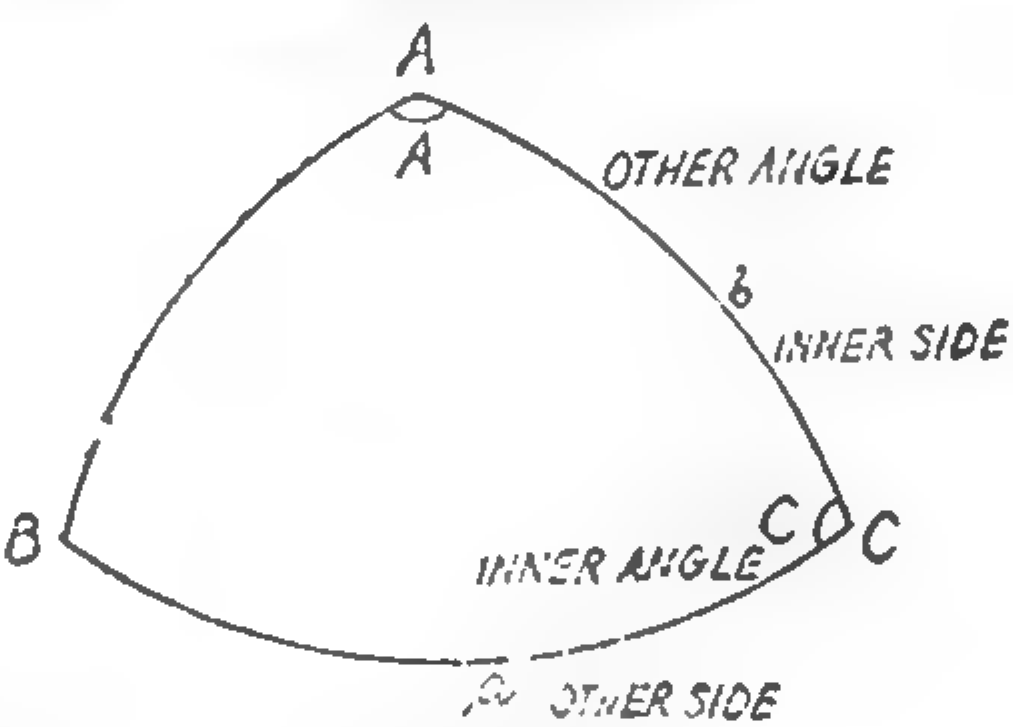


Fig. 18

cos (inner side) cos (inner angle)
=sin (inner side) cot (other side)
−sin (inner angle) cot (other angle)

i.e. $$\mathbf{\cos b \cos C = \sin b \cot a - \sin C \cot A.}$$

Proof.

$\cos a = \cos b \cos c + \sin b \sin c \cos A.$ ...(1)

[§ 1 R. 4].

$\cos c = \cos a \cos b + \sin a \sin b \cos C$

$\sin c = \dfrac{\sin a}{\sin A} \sin C.$ [§ 3]

In the formula to be proved we do not want the element c. Hence putting the values of $\cos c$ and $\sin c$ in (1), we get

$$\cos a = \cos b\,(\cos a \cos b + \sin a \sin b \cos C) + \sin b \cos A . \frac{\sin a}{\sin A} \sin C.$$

$$\therefore \quad \cos a\,(1 - \cos^2 b) = \sin a \sin b \cos b \cos C + \sin a \sin b \cot A \sin C.$$

Dividing throughout by $\sin a \sin b$, we get

$$\cot a \sin b = \cos b \cos C + \cot A \sin C$$

or $$\mathbf{\cos b \cos C = \sin b \cot a - \sin C \cot A.}$$

§ 9. **Napier's Analogies :—**

$$\tan \frac{A+B}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \cot \frac{C}{2} \qquad \ldots(1)$$

[Utkal 56, Vikram 59, Agra 61]

$$\tan \frac{A-B}{2} = \frac{\sin \frac{a-b}{2}}{\sin \frac{a+b}{2}} \cot \frac{C}{2} \qquad \ldots(2)$$

$$\tan \frac{a+b}{2} = \frac{\cos \frac{A-B}{2}}{\cos \frac{A+B}{2}} \tan \frac{c}{2}, \qquad \ldots(3)$$

Utkal 54]

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2} \qquad \ldots(4)$$

It is easy to observe that when we have got sum i.e. $\tan\frac{A+B}{2}$ *and* $\tan\frac{a+b}{2}$ *in the L.H.S. then we get cosine in the R.H.S; when we have differencs i.e.* $\tan\frac{A-B}{2}$ *and* $\tan\frac{a-b}{2}$ *in the L.H.S. then we get sine in the R.H.S. Result (2) is analogous with th result of plane trigonometry i.e.* $\tan\frac{B-C}{2}=\frac{b-c}{2}\cot\frac{A}{2}$.

Proof. 1st Method.

$$\tan\frac{A}{2}=\sqrt{\left[\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}\right]} \quad [\S\ 4,\ R.\ 2]$$

and $$\tan\frac{C}{2}=\sqrt{\left[\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}\right]}.$$

$$\therefore\quad \tan\frac{A}{2}\tan\frac{C}{2}=\frac{\sin(s-b)}{\sin s}.$$

Similarly $\tan\frac{B}{2}\tan\frac{C}{2}=\frac{\sin(s-a)}{\sin s}$ and

$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{\sin(s-c)}{\sin s}.$$

Now $$\tan\frac{A+B}{2}\tan\frac{C}{2}=\frac{(\tan A/2+\tan B/2)}{1-\tan A/2\tan B/2}\tan\frac{C}{2}$$

$$=\frac{\frac{\sin(s-b)}{\sin s}+\frac{\sin(s-a)}{\sin s}}{1-\frac{\sin(s-c)}{\sin s}}=\frac{\sin(s-b)+\sin(s-a)}{\sin s-\sin(s-c)}$$

or $$\tan\frac{A+B}{2}\tan\frac{C}{2}=\frac{2\sin\frac{2s-a-b}{2}\cos\frac{a-b}{2}}{2\sin\frac{s-s+c}{2}\cos\frac{s+s-c}{2}}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}$$

$[\because\ 2s=a+b+c]$

$$\therefore \quad \tan\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2} \text{ which proves (1).}$$

Similarly we can prove the result (2).

Again we know that

$$\tan\frac{a}{2}=\sqrt{\left(\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}\right)} \text{ [§ 5 Result 3]}$$

$$\therefore \quad \tan\frac{a}{2}\cot\frac{c}{2}$$

$$=\sqrt{\left\{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}\right\}}\sqrt{\left\{\frac{\cos(S-A)\cos(S-B)}{-\cos S\cos(S-C)}\right\}}$$

$$=\frac{\cos(S-A)}{\cos(S-C)}.$$

Similarly $\tan\frac{b}{2}\cot\frac{c}{2}=\frac{\cos(S-B)}{\cos(S-C)}$

and $\tan\frac{a}{2}\tan\frac{b}{2}=\frac{-\cos S}{\cos(S-C)}.$

Now $\tan\frac{a+b}{2}\cot\frac{c}{2}=\dfrac{\tan\frac{a}{2}+\tan\frac{b}{2}}{1-\tan\frac{a}{2}\tan\frac{b}{2}}\cot\frac{c}{2}.$

$$=\frac{\frac{\cos(S-A)}{\cos(S-C)}+\frac{\cos(S-B)}{\cos(S-C)}}{1+\frac{\cos S}{\cos(S-C)}}=\frac{\cos(S-A)+\cos(S-B)}{\cos(S-C)+\cos S}$$

or $$\tan\frac{a+b}{2}\cot\frac{c}{2}=\frac{2\cos\frac{2S-A-B}{2}\cos\frac{A-B}{2}}{2\cos\frac{S-C-S}{2}\cos\frac{S-C+S}{2}}$$

$$=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}.$$

$$\therefore \quad A+B+C=2S.$$

$$\therefore \quad \tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}.$$

Similarly we can prove that

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}.$$

Note. The results Nos. 3 and 4 can also be proved by replacing the angles by the supplements of sides and sides by supplements of angles in results Nos. 1 and 2.

Alternative Method.

$$\tan\frac{A+B}{2}=\frac{\sin\frac{A+B}{2}}{\cos\frac{A+B}{2}}\cdot\frac{2\cos\frac{A-B}{2}}{2\cos\frac{A-B}{2}}=\frac{\sin A+\sin B}{\cos A+\cos B}\ldots(1)$$

Now $\sin A+\sin B=K(\sin a+\sin b)$ [by sine formula (1)]

$\cos A=-\cos B\cos C+\sin B\sin C\cos a.$ [§ 2 (1)]

$\cos B=-\cos C\cos A+\sin C\sin A\cos b.$

$$\therefore \quad (\cos A+\cos B)=-\cos C(\cos A+\cos B)$$
$$+\sin C\,[K\sin b\cos a+K\sin a\cos b].$$

$$\therefore \quad (\cos A+\cos B)(1+\cos C)=\sin C.K\sin(a+b).$$

$$\therefore \quad \cos A+\cos B=\frac{K\sin C}{1+\cos C}\sin(a+b)$$

$$=K.\frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\cos^2\frac{C}{2}}\sin(a+b)$$

$$=K\tan\frac{C}{2}\sin(a+b).$$

Now put for $\sin A+\sin B$ and $\cos A+\cos B$ in (1).

$$\therefore \quad \tan\frac{A+B}{2}=\frac{K(\sin a+\sin b)}{K.\tan\frac{C}{2}\sin(a+b)}$$

$$=\cot\frac{C}{2}.\ \frac{2\sin\frac{a+b}{2}\cos\frac{a-b}{2}}{2\sin\frac{a+b}{2}\cos\frac{a+b}{2}}.$$

$$\therefore \quad \tan\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2}.$$

Similarly we can prove the other analogies.

§ 10. Delambre's Analogies.

$$\frac{\sin\frac{A+B}{2}}{\sin\frac{C}{2}}=\frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}} \qquad \ldots(1)$$

(Nagpur 61)

$$\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}=\frac{\sin\frac{a-b}{2}}{\sin\frac{c}{2}} \qquad \ldots(2)$$

$$\frac{\cos\frac{A+B}{2}}{\sin\frac{C}{2}}=\frac{\cos\frac{a+b}{2}}{\cos\frac{c}{2}} \qquad \ldots(3)$$

and

$$\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}=\frac{\sin\frac{a+b}{2}}{\sin\frac{c}{2}}. \qquad \ldots(4)$$

(Utkal 54, Nagpur 56, 57)

$$\sin\frac{A+B}{2}=\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}.$$

$$=\sqrt{\left[\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}\right]}\sqrt{\left[\frac{\sin s\sin(s-b)}{\sin a\sin c}\right]}$$

$$+\sqrt{\left[\frac{\sin s\sin(s-a)}{\sin b\sin c}\right]}\sqrt{\left[\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}\right]}$$

$$=\left[\frac{\sin(s-b)}{\sin c}+\frac{\sin(s-a)}{\sin c}\right]\sqrt{\left[\frac{\sin s\sin(s-c)}{\sin a\sin b}\right]}$$

$$=\frac{2\sin\frac{2s-a-b}{2}\cos\frac{a-b}{2}}{2\sin\frac{c}{2}\cos\frac{c}{2}}\cos\frac{C}{2}.$$

$$\therefore\quad \frac{\sin\frac{A+B}{2}}{\cos\frac{C}{2}}=\frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}\quad\because\quad 2s=a+b+c.$$

Similarly we can prove the other results.

Exercise 1

1. (a) *If D be the middle point of AB, show that*

$$\cos AC+\cos BC=2\cos(AB/2)\cos CD,$$

i. e.

$$\cos a+\cos b=2\cos(c/2)\cos CD.$$

(Dacca 51, Agra 55)

Join C and D and let $\angle ADC=\alpha$, so that $\angle BDC=\pi-\alpha$. Now applying cosine formula on triangles ACD and BCD, we get $AD=BD=c/2$ as D is mid point of AB.

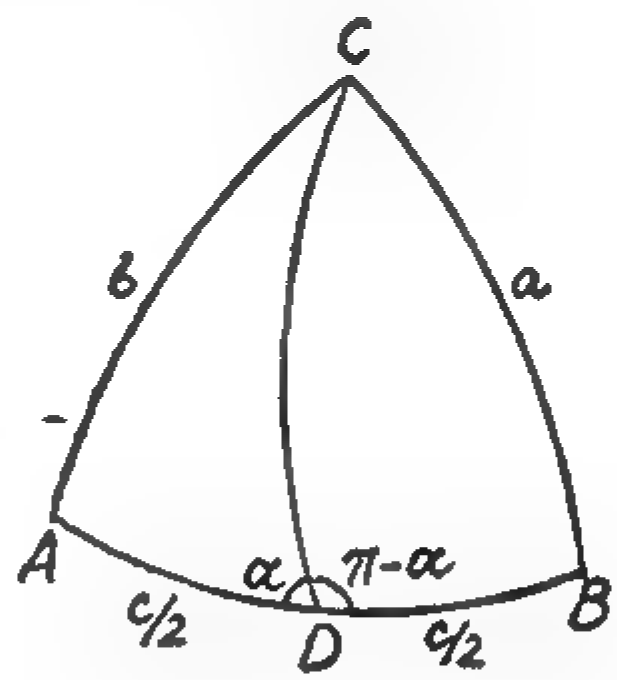


Fig. 19

$$\cos b=\cos(c/2)\cos CD+\sin(c/2)\sin CD\cos\alpha$$

and $\cos a=\cos(c/2)\cos CD+\sin(c/2)\sin CD\cos(\pi-\alpha)$.

Adding, we get

$$\cos a+\cos b=2\cos(c/2)\cos CD,\quad\because\quad \cos(\pi-\alpha)=-\cos\alpha.$$

(b) *If two sides of a spherical triangle be supplementary, prove that the median passing through their intersection is a quadrant.*

Let us choose that AC and BC are suplementary so that $a+b=\pi$ and we have to prove that the median passing through their intersection C, *i. e.* $CD=\pi/2$.

$$\cos a+\cos b=2\cos(c/2)\cos CD \qquad \text{[by part (a)].}$$

But $\cos a=\cos(\pi-b)=-\cos b$ or $\cos a+\cos b=0$.

$$\therefore \quad 2\cos(c/2)\cos CD=0 \quad \text{or} \quad \cos CD=0.$$

$$\therefore \quad CD=\pi/2.$$

Note. $\cos(c/2)$ cannot be zero for in that case $c/2=\pi/2$ or $c=\pi$ which is not possible.

We have proved above that $CD=\pi/2$ when $a+b=\pi$ (given).

$\therefore$ $a+b=2CD$ showing that CD is the arithmetic mean of AC and BC, *i. e.* CA, CD and CB are in arithmetical progression.

Another form.

If D be the middle point of the base AB of a spherical triangle ABC and CA, CD and CB be in arithmetical progression, show that CD is a quadrant.

We are given that $2CD=a+b$.

(c) *If one angle of a spherical triangle be equal to the sum of the other two, the greater side is double of the distance of its middle point from the vertex.* **(Delhi 54)**

Let us suppose that $C=A+B$ and we have to prove that greater side AB is double of median CD *i. e.* $AB=2CD$.

We have proved in part (a) that

$$\cos a+\cos b=2\cos(c/2)\cos CD. \qquad \ldots(1)$$

Again $\cos a=\cos b\cos c+\sin b\sin c\cos A$ [cosine formula].

$$\cos b=\cos c\cos a+\sin c\sin a\cos B.$$

$$\therefore \quad \cos a+\cos b=\cos c\,(\cos a+\cos b) + \sin c\,.K\,[\sin A\cos B+\cos A\sin B].$$

$$\therefore \quad \frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}=K \text{ say.} \qquad \ldots(2)$$

$$\therefore \quad (\cos a+\cos b)(1-\cos c)=\sin c . K \sin (A+B)$$
$$=\sin c . K \sin C,$$
$$\because \quad A+B=C \text{ given}$$

or $\quad 2 \cos (c/2) \cos CD.(1-\cos c)=\sin^2 c=1-\cos^2 c$

[by (1) and (2)]

or $\quad 2 \cos (c/2) \cos CD=1+\cos c=2 \cos^2 c/2.$

$$\therefore \quad \cos CD=\cos c/2 \quad \text{or} \quad CD=c/2=AB/2.$$

$$\therefore \quad 2CD=AB.$$ **Hence proved.**

Alternative Method.

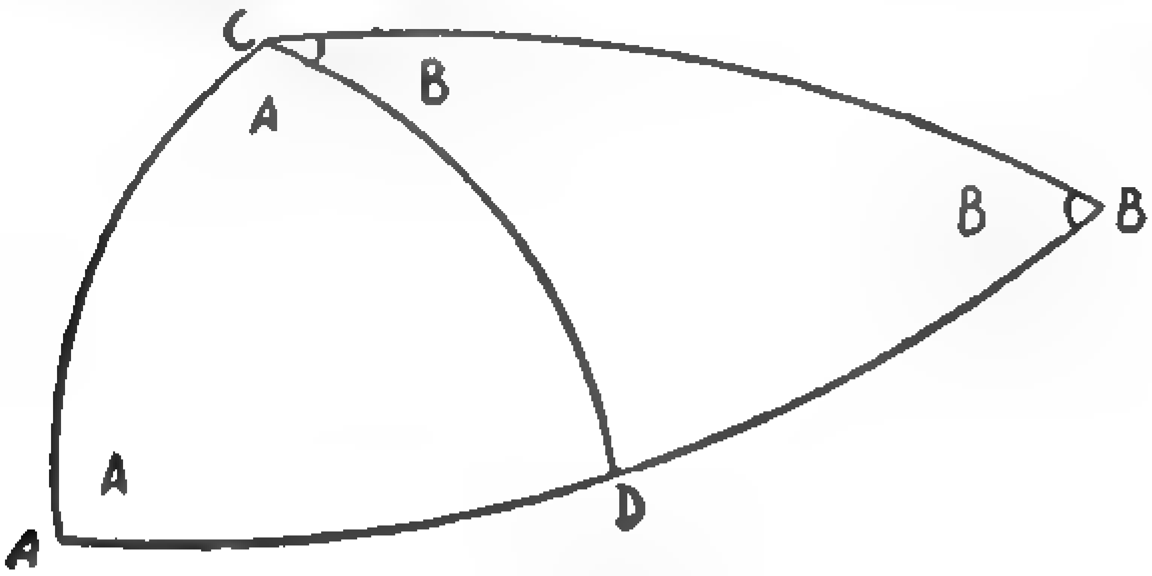


Fig. 20

We are given that $\angle C=\angle A+\angle B$. Through C draw CD such that $\angle ACD=\angle A$; therefore $\angle DCB=B$. Hence we have $AD=CD$ and $BD=CD$, showing that D is mid. point of AB and also $2CD=AD+BD=AB$. **Hence proved.**

(d) *If $C=A+B$, prove that the middle point of AB is equidistant from the three vertices of the triangle ABC.*

(d_1) *If D be any point BC, prove that*

(i) $\cos AD \sin BC=\cos AB \sin CD+\cos AC \sin BD$,

(Nagpur 56, Utkal 52, Agra 60, Delhi 60, Jabalpur 60)

(ii) $\cot AD \sin A=\cot c \sin DAC+\cot b \sin BAD$.

(Agra 57, Delhi 60)

(i) Let us choose that $\angle BDA = \alpha$, so that $\angle CDA = \pi - \alpha$.

$\cos AB = \cos BD \cos AD + \sin BD \sin AD \cos \alpha \ldots (1)$

$\cos AC = \cos CD \cos AD + \sin CD \sin AD \cos (\pi - \alpha) \ldots (2)$

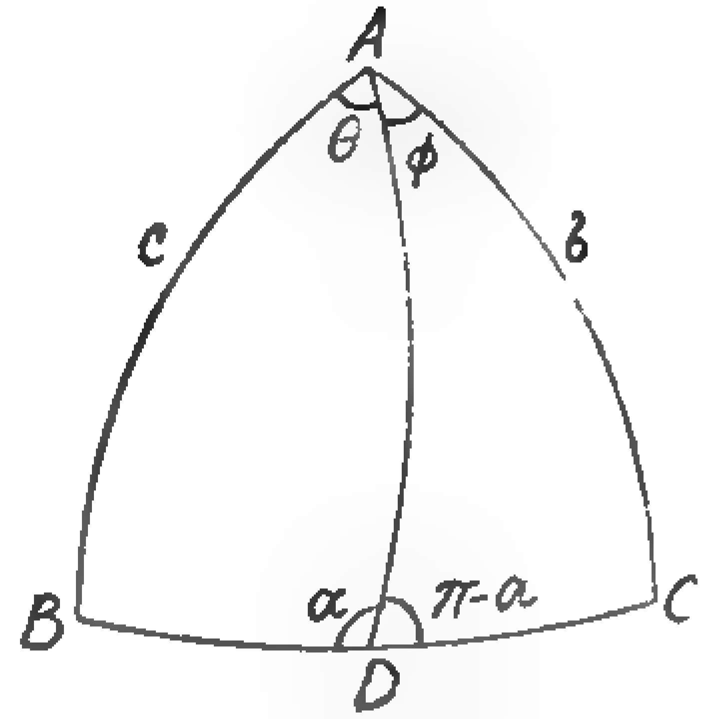


Fig. 21

In order to eliminate α we multiply (1) by $\sin CD$ and (2) by $\sin BD$ and add.

$$\therefore \quad \cos AB \sin CD + \cos AC \sin BD$$
$$= \cos AD (\sin CD \cos BD + \cos CD \sin BD)$$
$$= \cos AD \sin (CD + BD) = \cos AD \sin BC.$$

(ii) Let $\angle BAD = \theta$ and $\angle DAC = \phi$, so that $\theta + \phi = A$.

Since the relation to be proved involves cotangent, we apply the formula of consecutive four on Δs ABD and ADC which is cos (inner side).cos (inner angle)

= sin (inner side) cot (other side)

− sin (inner angle) cot (other angle).

In ΔABD choosing the consecutive four elements to be AB, θ, AD and α, where θ is the inner angle and AD the inner side,

$$\cos AD \cos \theta = \sin AD \cot AB - \sin \theta \cot \alpha \quad \ldots (3)$$
[from ΔABD]

and $$\cos AD \cos \phi = \sin AD \cot AC - \sin \phi \cot (\pi - \alpha) \ldots (4)$$
[from ΔACD]

In order to eliminate α, multiply (3) by $\sin \phi$ and (4) by $\sin \theta$ and add, keeping in view that $\cot (\pi - \alpha) = -\cot \alpha$.

$$\therefore \quad \cos AD [\sin \phi \cos \theta + \cos \phi \sin \theta]$$
$$= \sin AD (\sin \phi \cot AB + \sin \theta \cot AC)$$

or $$\cot AD \sin (\theta + \phi) = \cot c \sin DAC + \cot b \sin DAB$$

or $$\cot AD \sin A = \cot c \sin DAC + \cot b \sin DAB.$$

Hence proved.

2. (a) *In a spherical triangle ABC, if θ be the arc bisecting the angle A and terminated by the opposite side, prove that*

$$2 \cot \theta \cos \frac{A}{2} = \cot b + \cot c.$$

(Benares 57, Sagar 52)

Let $\angle BDA = \alpha$ so that $\angle ADC = \pi - \alpha$. Since the relation to be proved involves cotangent, let us apply the formula of consecutive four as explained in last question on triangles ABD and ACD. The four consecutive elements to be chosen are marked in the figure and $AD = \theta$.

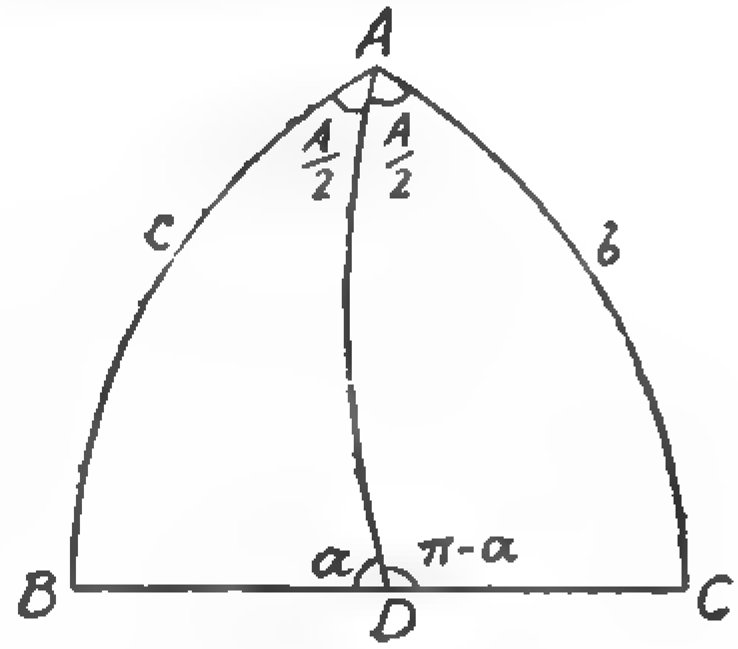


Fig. 22

$$\cos \theta \cos \frac{A}{2} = \sin \theta \cot c - \sin \frac{A}{2} \cot \alpha \text{ from } \triangle ABD.$$

and $\cos \theta \cos \dfrac{A}{2} = \sin \theta \cot b - \sin \dfrac{A}{2} \cot (\pi - \alpha)$ from $\triangle ACD$.

Adding, we get

$$2 \cos \theta \cos \frac{A}{2} = \sin \theta (\cot b + \cot c),$$

$$\because \cot (\pi - \alpha) = -\cot \alpha$$

or $\qquad 2 \cot \theta \cos \dfrac{A}{2} = \cot b + \cot c.$ $\qquad$ Proved.

(b) *In a spherical triangle, if θ, ϕ, ψ if be the arcs bisecting the angles A, B, C respectively and terminated by the opposite sides, show that*

$$\cot \theta \cos \frac{A}{2} + \cot \phi \cos \frac{B}{2} + \cot \psi \cos \frac{C}{2}$$
$$= \cot a + \cot b + \cot c.$$

(Agra 59, Gorakhpur 60, Gujarat 56, Utkal 56, 59, Nagpur 56, 61, Lucknow 52)

Write three similar relations as in part (a) and add.

(c) *In a spherical triangle if δ be the arc of external bisector of $\angle A$ and terminated by opposite side, prove that*

$$2 \cot \delta_1 \sin \frac{A}{2} = \cot b - \cot c.$$

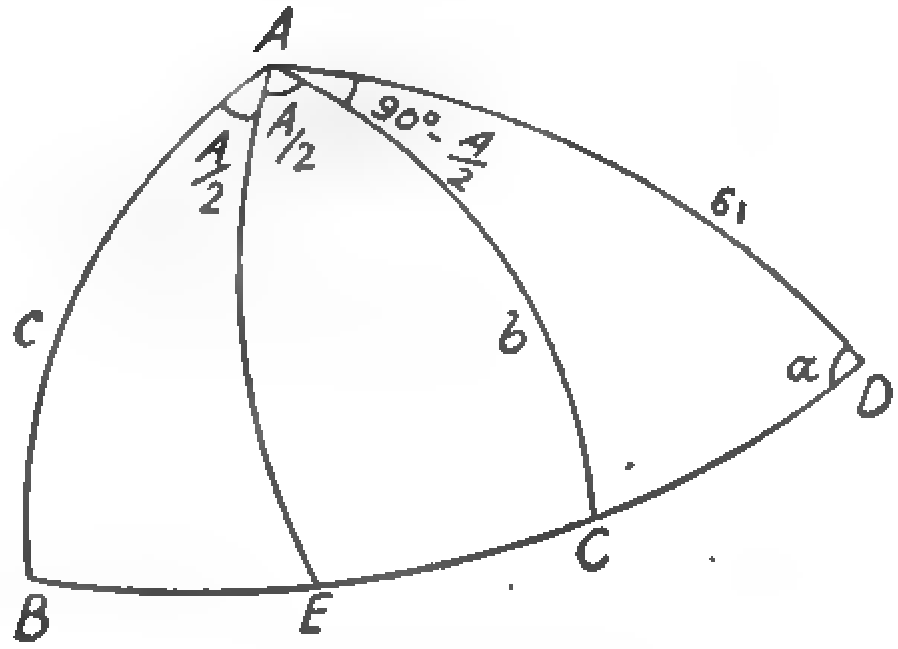


Fig. 23

Let AD be external bisector of angle A which is perpendicular to AE the internal bisector.

$\therefore \quad \angle DAC = 90 - A/2$

and $\quad \angle DAB = 90 + A/2.$

Let $\angle BDA = \alpha.$

Applying the formula of consecutive four on triangles DAC and DAB, we get

$\cos \delta_1 \cos (90 - A/2) = \sin \delta_1 \cot b - \sin (90 - A/2) \cot \alpha$ from $\triangle DAC$

$\cos \delta_1 \cos (90 + A/2) = \sin \delta_1 \cot c - \sin (90 + A/2) \cot \alpha$ from $\triangle DAB.$

Subtracting, keeping in view that $\cos (90 + A/2) = -\sin A/2$, we get

$$2 \cos \delta_1 \sin A/2 = \sin \delta_1 (\cot b \cdot \cot c)$$

or

$$2 \cot \delta_1 \sin A/2 = \cot b - \cot c.$$

(d) *In a spherical triangle if $\delta_1, \delta_2, \delta_3$ be the arcs of the external bisectors of the angles A, B and C of a spherical triangle ABC and terminated by the opposite sides, prove that*

$\cot \delta_1 \sin A/2 + \cot \delta_2 \sin B/2 + \cot \delta_3 \sin C/2 = 0.$ **(Nagpur 57)**

Write three relations as is part (c) and add.

(e) *If AX and AY be the internal and external bisectors of A, which is a right angle, prove that*

$$\cot^2 AX + \cot^2 AY = \cot^2 b + \cot^2 c.$$

(Dehli 59, Benares 48)

Since $\angle A = \pi/2$, $\therefore A/2 = \pi/4$; $\therefore \sin A/2 = \cos A/2 = 1\sqrt{2}.$

Hence from parts (a) and (c), we get

$$2 \cot AX . \cos \pi/4 = \cot b + \cot c$$

and
$$2 \cot AY \sin \pi/4 = \cot b - \cot c.$$

Squaring and adding,

$$4 (\cot^2 AX + \cot^2 AY) . \tfrac{1}{2} = 2 (\cot^2 b + \cot^2 c) \text{ etc.}$$

(f) *If α and β denote the angles which the internal and external bisectors of angle A make with the side BC, show that*

$$2 \cos (A/2) \cos \alpha = \cos B - \cos C,$$
$$2 \sin (A/2) \cos \beta = \cos B + \cos C.$$

The relations to be proved involve all the angles; we use the formula of supplemental cosines [§ 2],

i.e. $\cos A = -\cos B \cos C + \sin B \sin C \cos a$

or $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$

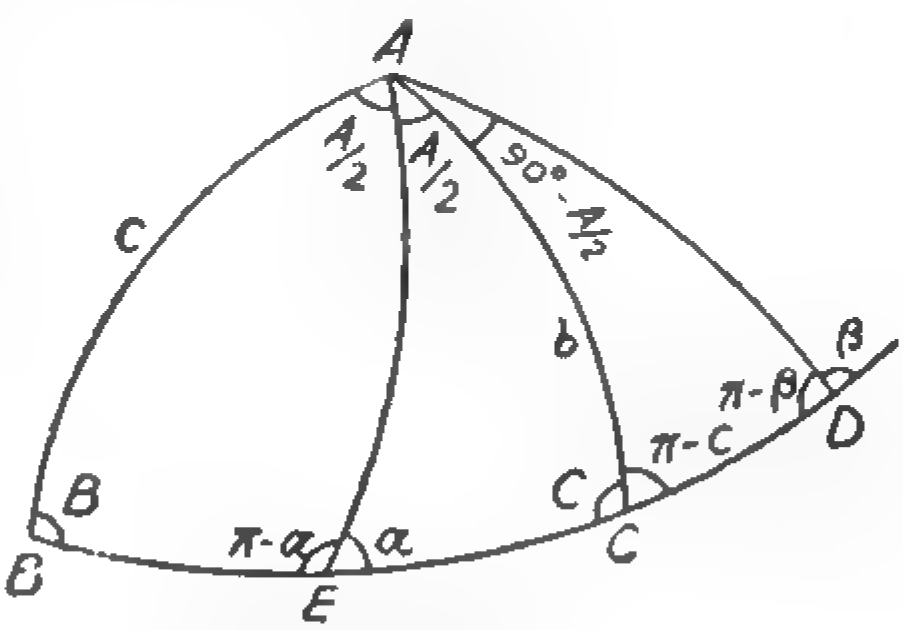


Fig. 24

Applying the above formula on Δs ABE and ACE and equating the values of $\cos AE$, we get

$$\frac{\cos B + \cos A/2 \cos (\pi - \alpha)}{\sin A/2 \sin (\pi - \alpha)} = \frac{\cos C + \cos A/2 \cos \alpha}{\sin A/2 \sin \alpha} = \cos AE$$

$$\cos (\pi - \alpha) = -\cos \alpha \text{ and } \sin (\pi - \alpha) = \sin \alpha.$$

$$\therefore \quad \cos B - \cos (A/2) \cos \alpha = \cos C + \cos (A/2) \cos \alpha.$$

$$\therefore \quad 2 \cos (A/2) \cos \alpha = \cos B - \cos C$$

which proved first part.

Similarly applying the same formula on Δs ACD and ABD and equating the values of $\cos AD$, we get

$$\frac{\cos (\pi - C) + \cos (90 - A/2) \cos (\pi - \beta)}{\sin (90 - A/2) \sin (\pi - \beta)}$$

$$\frac{\cos B + \cos (90 + A/2) \cos (\pi - \beta)}{\sin (90 + A/2) \sin (\pi - \beta)} = \cos AD$$

$\cos(\pi - C) = -\cos C$ and $\cos(90 + A/2) = -\sin A/2$,

$\sin(90 \pm A/2) = \cos A/2$.

$\therefore \quad -\cos C - \sin(A/2)\cos\beta = \cos B + \sin(A/2)\cos\beta.$

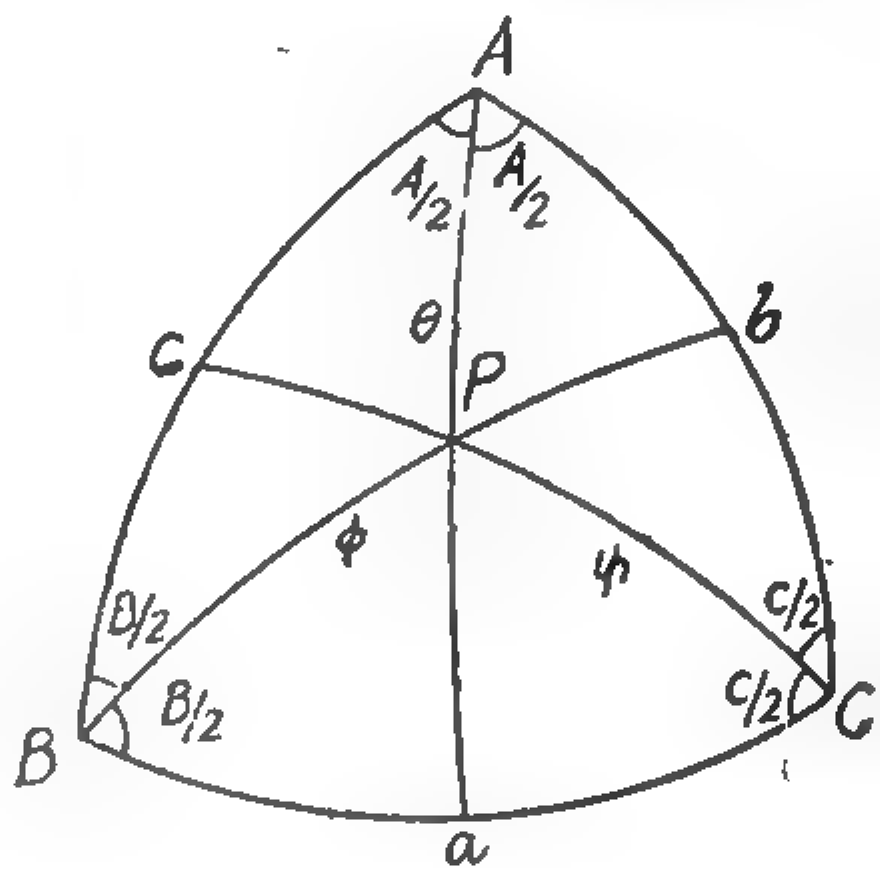
$\because \quad 2\sin A/2\cos\beta = -(\cos B + \cos C).$

Note. The $-$ive sign will disappear if we choose β in place of $\pi - B$.

(g) *If θ, ϕ, ψ, denote the distances of the corners A, B, C respectively from the point of intersection of the arcs bisecting the angles of the spherical triangle ABC. show that*

$$\cos\theta\sin(b-c) + \cos\phi\sin(c-a) + \cos\psi\sin(a-b) = 0.$$

(Utkal 54)



Applying the formula of concecutive four on $\triangle$s APB and APC, we get

$\cos c\cos A/2 = \sin c\cot\theta - \sin A/2\cot B/2 \ldots.(1)$

$\cos b\cos A/2 = \sin b\cot\theta - \sin A/2\cot C/2 \ldots(2)$

Multiply (2) by $\cos c$ and (1) by $\cos b$ and subtract.

$\therefore \quad 0 = \cot\theta(\sin b\cos c - \cos b\cos c) - \sin A/2[\cos c\cot C/2 - \cos b\cot B/2]$

or $\quad \cos\theta\sin(b-c) = \sin A/2\sin\theta[\cos c\cot C/2 - \cos b\cot B/2] \ldots(1)$

Again by sine formula we have

$$\frac{\sin\theta}{\sin B/2} = \frac{\sin\phi}{\sin A/2}$$

or $\quad \sin\theta\sin A/2 = \sin\phi\sin B/2 = \sin\psi\sin C/2 = k$ say by symmetry.

The relation (1) by the help of above relation becomes

$\cos\theta\sin(b-c) = k[\cos c\cot C/2 - \cos b\cot B/2]. \quad \ldots(2)$

Again by symmetry we can write the other relation as under :—

$\cos\phi \sin(c-a)=k[\cos a \cot A/2-\cos c \cot C/2]$. ...(3)

$\cos\psi \sin(a-b)=k[\cos b \cot B/2-\cos a \cot A/2]$. ...(4)

Adding (2), (3) and (4), we get

$\cos\theta \sin(b-c)+\cos\phi \sin(c-a)+\cos\psi \sin(a-b)=0.$

Proved.

3. (a) *In a spherical triangle, if θ, ϕ, ψ be the arcs of great circles drawn from A, B, C perpendicular to the opposite sides, then prove that $\sin a \sin\theta=\sin b \sin\phi=\sin c \sin\psi$*

$$=\{1-\cos^2 a-\cos^2 b-\cos^2 c+2 \cos a \cos b \cos c\}^{1/2}.$$

[Nagpur 58, Alld. 52, 58]

Applying sine formula on $\triangle ABD$ or $\triangle ADC$, we get

$$\frac{\sin\theta}{\sin B}=\frac{\sin c}{\sin\frac{\pi}{2}}, \quad \therefore \quad \sin\theta=\sin c \sin B.$$

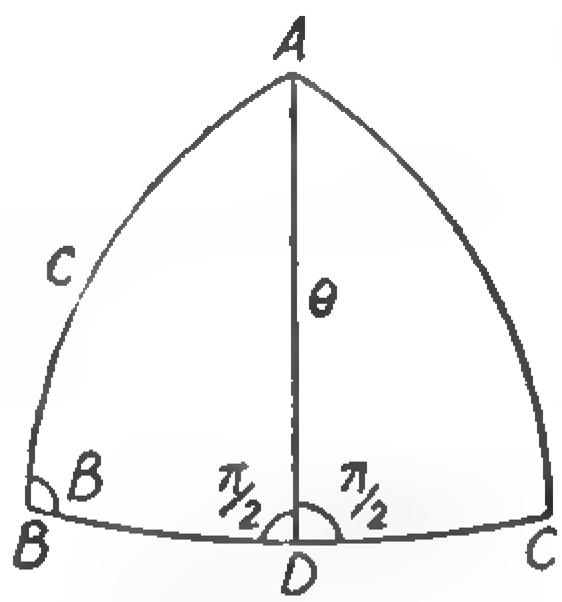


Fig. 25.

$$\therefore \quad \sin a \sin\theta=\sin a \sin c \sin B$$

$$=\sin a \sin c \frac{2n}{\sin a \sin c}$$

or $\sin a \sin\theta=2n=\{1-\cos^2 a-\cos^2 b-\cos^2 c$
$+2 \cos a \cos b \cos c\}^{1/2}$. [§ 3]

The symmetry of the result proves other results.

(a$_1$) *If δ be the length of arc drawn from C perpendicular to AB in any spherical triangle ABC, show that*

$$\cos^2 a+\cos^2 b-2 \cos a \cos b \cos c=\sin^2 c \cos^2 \delta$$

or $\cos\delta=\operatorname{cosec} c\ [\cos^2 a+\cos^2 b-2 \cos a \cos b \cos c]^{1/2}$.

[Alld. 52, Benares 57, Bihar 60, Vikram 59, Raj. 58]

From part (a), we have $\sin\delta \sin C=2n$.

$$\therefore \quad \cos^2\delta=1-\sin^2\delta=1-\frac{4n^2}{\sin^2 c}$$

$$\therefore \quad \sin^2 c \cos^2 \delta = \sin^2 c - (1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)$$
$$= \cos^2 a + \cos^2 b - 2 \cos a \cos b \cos c, \qquad \because \quad \sin^2 c + \cos^2 c - 1 = 0.$$

(b) *If $A=a$, show that B and b are equal or supplemental as also C and c.*

$\dfrac{\sin A}{\sin a}=\dfrac{\sin B}{\sin b}=\dfrac{\sin C}{\sin c}$. Now $A=a$. $\therefore$ $\sin A=\sin a$.

$$\therefore \quad \sin B = \sin b \quad \therefore \quad B = b \quad \text{or} \quad \pi - b.$$

Similarly $C=c$ or $\pi - c$.

(c) *If $b+c=\pi$, show that $\sin 2B+\sin 2C=0$.* **(Alld. 52)**

$$\sin 2B+\sin 2C = 2 \sin B \cos B + 2 \sin C \cos C$$
$$=2k\left[\sin b\,\frac{\cos b-\cos a\cos c}{\sin a \sin c}+\sin c\,\frac{\cos c-\cos a \cos b}{\sin a \sin b}\right]$$

Now $b=\pi-c$, $\therefore$ $\sin b=\sin c$ and $\cos b=-\cos c$.

$\therefore$ $\sin 2B+\sin 2C$

$$=\frac{2k}{\sin a}[-\cos c-\cos a\cos c+\cos c+\cos a \cos c]=0.$$

(d) *In any spherical triangle, prove that*

$$\tan\frac{A-a}{2}\tan\frac{B+b}{2}=\tan\frac{B-b}{2}\tan\frac{A+a}{2}.$$

(Nagpur 57)

By sine formula, we know that $\dfrac{\sin A}{\sin a}=\dfrac{\sin B}{\sin b}$,

$$\therefore \quad \frac{\sin A-\sin a}{\sin A+\sin a}=\frac{\sin B-\sin b}{\sin B+\sin b}$$

[by componendo and dividendo]

or
$$\frac{2\sin\frac{A-a}{2}\cos\frac{A+a}{2}}{2\sin\frac{A+a}{2}\cos\frac{A-a}{2}}=\frac{2\sin\frac{B-b}{2}\cos\frac{B+b}{2}}{2\sin\frac{B+b}{2}\cos\frac{B-b}{2}} \text{ etc.}$$

(e) *In any spherical triangle, prove that*

$$\frac{\sin (A+B)}{\sin C}=\frac{\cos a+\cos b}{1+\cos c} \text{ and } \frac{\sin (a+b)}{\sin c}=\frac{\cos A+\cos B}{1-\cos C}.$$

(Nagpur 61, Rajputana 52, Sagar 57, Agra 56, 58, Punjab 52)

$$\text{L.H.S.}=\frac{\sin A}{\sin C}\cos B+\frac{\sin B}{\sin C}\cos A$$

$$=\frac{\sin a}{\sin c}\,\frac{\cos b-\cos a\cos c}{\sin a\sin c}+\frac{\sin b}{\sin c}\,\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

[by sine formula]

$$=\frac{(\cos a+\cos b)-\cos c\,(\cos a+\cos b)}{\sin^2 c}$$

$$=\frac{(\cos a+\cos b)\,(1-\cos c)}{(1-\cos^2 c)}=\frac{\cos a+\cos b}{1+\cos c}.$$

For 2nd part change $\cos a$ and $\cos b$ in terms of cosine of angles etc.

(f) *In any spherical triangle, prove that*

$$2\cos\frac{a+b}{2}\cos\frac{a-b}{2}\tan\frac{c}{2}=\sin b\cos A+\sin a\cos B.$$

(Rajputana 54)

Put the values of $\cos A$ and $\cos B$ in R. H. S.

(g) *If $C=A+B$, prove that $1-\cos a-\cos b+\cos c=0$.*

(Nagpur 57)

If $a+b=\pi+c$, prove that $1+\cos A+\cos B-\cos C=0$.

It follows from part (e).

(g_1) *In a spherical triangle ABC, b and c are compementary; prove that $(\cos c+\sin c)\sin A=2\cos^2\frac{a}{2}\sin(B+C)$.*

From part (e), $\dfrac{\sin(B+C)}{\sin A}=\dfrac{\cos b+\cos c}{1+\cos a}=\dfrac{\sin c+\cos c}{2\cos^2\frac{a}{2}}$

$\therefore\quad b=\frac{\pi}{2}-c,\quad \therefore\quad \cos b=\sin c.$

(h) *In any spherical triangle, prove that*

(i) $\dfrac{\sin a}{\sin A}=\sqrt{\left(\dfrac{1-\cos a\cos b\cos c}{1+\cos A\cos B\cos C}\right)}.$

(Agra 48)

(ii) $\dfrac{\sin^2 a+\sin^2 b+\sin^2 c}{\sin^2 A+\sin^2 B+\sin^2 C}=\dfrac{1-\cos a\cos b\cos c}{1+\cos A\cos B\cos C}.$

(i) $\dfrac{\sin a}{\sin A}=\dfrac{\sin b}{\sin B}=\dfrac{\sin c}{\sin C}=k$ say.

Now $\cos a=\cos b\cos c+\sin b\sin c\cos A.$

Multiply by $\cos a$.

$\therefore\quad \cos^2 a-\sin b\sin c\cos a\cos A=\cos a\cos b\cos c.$

$\therefore\quad 1-\cos a\cos b\cos c=1-\cos^2 a+\sin b\sin c\cos A\cos a$

$$=\sin^2 a+\sin b\sin c\cos A\cos a \quad \ldots(1)$$

Again $\cos A=-\cos B\cos C+\sin B\sin C\cos a.$

Multiply by $\cos A$.

$\therefore\quad \cos^2 A-\sin B\sin C\cos a\cos A=-\cos A\cos B\cos C.$

$\therefore\quad 1+\cos A\cos B\cos C$

$$=1-\cos^2 A+\sin B\sin C\cos a\cos A$$

$$=\sin^2 A+\sin B\sin C\cos a\cos A$$

$$=\frac{1}{k^2}(\sin^2 a+\sin b\sin c\cos a\cos A)$$

by sine formula. $\ldots(2)$

Dividing (1) and (2), we get

$$\frac{1-\cos a\cos b\cos c}{1+\cos A\cos B\cos C}=k^2=\frac{\sin^2 a}{\sin^2 A}\text{ etc.} \quad \ldots(3)$$

(ii) $\dfrac{\sin^2 a+\sin^2 b+\sin^2 c}{\sin^2 A+\sin^2 B+\sin^2 C}=k^2\dfrac{(\sin^2 A+\sin^2 B+\sin^2 C)}{(\sin^2 A+\sin^2 B+\sin^2 C)}$

$$=k^2\text{ etc.}$$

Hence from (3), L. H. S.=R. H. S.

(i) *If one side of a spherical triangle be divided into four equal parts and θ_1, θ_2, θ_3, θ_4 be the angles subtended at the opposite corner by the parts taken in order, show that*

$$\sin(\theta_1+\theta_2)\sin\theta_2\sin\theta_4=\sin(\theta_3+\theta_4)\sin\theta_1\sin\theta_3.$$

(Lucknow 1952, Agra 54, Rajputana 55, Jabalpur 60)

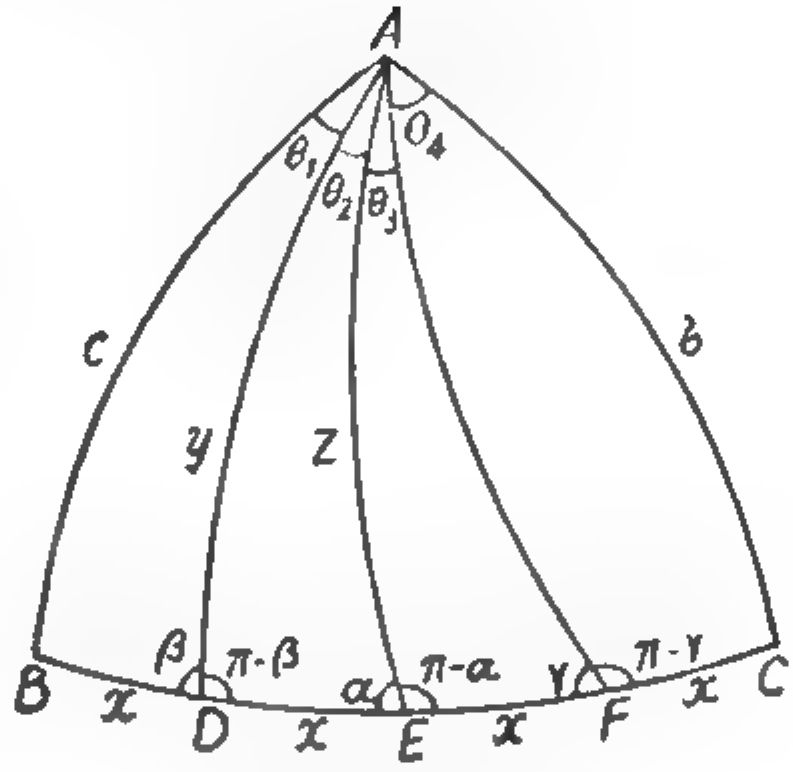


Fig. 23

Let the side BC be $4x$, so that $BD=DE=EF=FC=x$ and let $\angle AEB=\alpha$, so that $\triangle AEC=\pi-\alpha$ and $AD=y$ and $\angle ADB=\beta$ and $AE=z$.

We shall use sine formula on various triangles.

1. $\dfrac{\sin(\theta_1+\theta_2)}{\sin 2x}=\dfrac{\sin\alpha}{\sin c}$ from $\triangle ABE$.

2. $\dfrac{\sin(\theta_3+\theta_4)}{\sin 2x}=\dfrac{\sin(\pi-\alpha)}{\sin b}$ from $\triangle ACE$.

3. $\dfrac{\sin\theta_2}{\sin x}=\dfrac{\sin(\pi-\beta)}{\sin z}$ from $\triangle ADE$.

4. $\dfrac{\sin\theta_1}{\sin x}=\dfrac{\sin\beta}{\sin c}$ from $\triangle ABD$.

5. $\dfrac{\sin\theta_4}{\sin x}=\dfrac{\sin(\pi-\gamma)}{\sin b}$ from $\triangle AFC$.

6. $\dfrac{\sin\theta_3}{\sin x}=\dfrac{\sin\gamma}{\sin z}$ from $\triangle AEF$.

Multiplying 1, 3, 5, and 2, 4, 6, we get the required result.

(i_1) *If arcs be drawn from the angles of a spherical triangle to meet the middle points of the opposite sides and if α, β be the parts of arc which bisects the side a, show that $\dfrac{\sin\alpha}{\sin\beta}=2\cos\dfrac{a}{2}$.*

(Utkal 55, 59)

By sine formula we have the following :

(1) $\dfrac{\sin\alpha}{\sin\psi}=\dfrac{\sin b/2}{\sin\theta}$ from $\triangle AGE$.

(2) $\dfrac{\sin\beta}{\sin\phi}=\dfrac{\sin a/2}{\sin\theta}$ from $\triangle BGD$.

$$\therefore\quad \frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\phi}{\sin\psi}=\frac{\sin b/2}{\sin a/2} \quad \ldots(1)$$

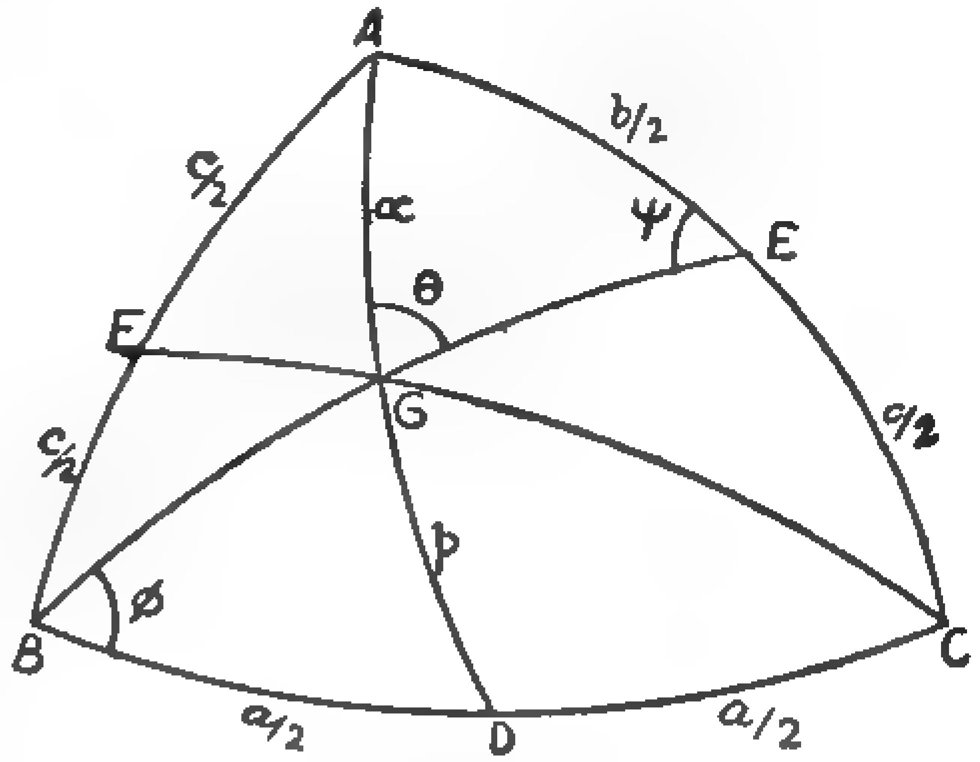


Fig. 27

Now from $\triangle BCE$, we get

$$\frac{\sin a}{\sin(\pi-\psi)}=\frac{\sin b/2}{\sin\phi};$$

$$\therefore\quad \frac{\sin\phi}{\sin\psi}=\frac{\sin b/2}{\sin a}.$$

Putting in (1), we get

$$\frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin b/2}{\sin a}=\frac{\sin b/2}{\sin a/2},$$

$$\therefore\quad \frac{\sin\alpha}{\sin\beta}=\frac{\sin a}{\sin a/2}=2\cos\frac{a}{2}.$$

(j) *If equal sides of an isosceles triangle ABC be bisected by an arc DE and BC be the base, show that*

$$\sin\frac{DE}{2}=\tfrac{1}{2}\sin\frac{BC}{2}\sec\frac{AC}{2}.$$

(Raj. 59, Benares 52)

We know that the line joining the mid. points of the sides of a triangle is parallel to the base and half its length. Also in the case of an isosceles triangle the median is perpendicular to the base.

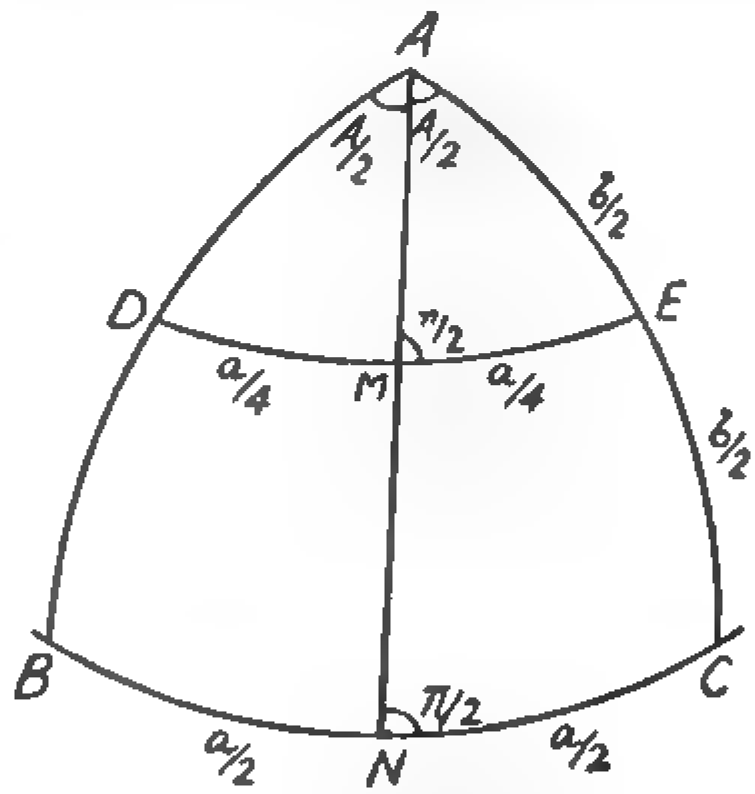


Fig. 28

Now by sine formula, we have

$$\frac{\sin a/4}{\sin A/2}=\frac{\sin b/2}{\sin \pi/2}=\sin b/2$$

from $\triangle AME$,

$$\frac{\sin a/2}{\sin A/2}=\frac{\sin b}{\sin \pi/2}=\sin b \qquad \text{from } \triangle ANC$$

or $\dfrac{\sin a/2}{\sin a/4}=\dfrac{\sin b}{\sin b/2}$ or $\dfrac{\sin a/2}{\sin a/4}=\dfrac{2 \sin b/2 \cos b/2}{\sin b/2}=2 \cos b/2$

or $$\sin \frac{BC}{2}=2 \sin \frac{DE}{2} \cos \frac{AC}{2}.$$

(k) *If E and F are the middle points of the sides AC, AB of a spherical triangle ABC and FE proucet meets BC produced in D, prove that*

$$\sin DE \cos \frac{b}{2}=\sin DF \cos \frac{c}{2}.$$

$$\frac{\sin DE}{\sin (\pi-C)}=\frac{\sin b/2}{\sin D}$$

$$\frac{\sin DF}{\sin B}=\frac{\sin c/2}{\sin D};$$

$$\therefore \quad \frac{\sin DE}{\sin DF}=\frac{\sin C}{\sin B}\cdot\frac{\sin b/2}{\sin c/2}$$

$$=\frac{\sin c}{\sin b}\cdot\frac{\sin b/2}{\sin c/2}$$

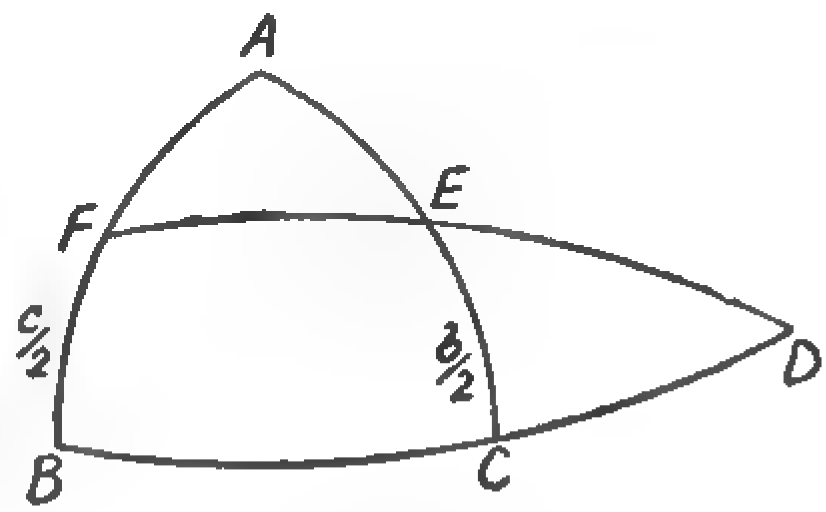


Fig. 29

by sine formula

$$=\frac{2 \sin c/2 \cos c/2}{2 \sin b/2 \cos b/2}\cdot\frac{\sin b/2}{\sin c/2}=\frac{\cos c/2}{\cos b/2}.$$

Hence proved.

(l) *In a spherical triangle ABC, D is the middle point of AB; prove that*

$$\cot BCD-\cot ACD=\frac{\sin^2 A-\sin^2 B}{\sin A \sin B \sin C}=\frac{\sin (a+b) \sin (a-b)}{\sin a \sin b \sin C}$$

(Agra 54)

Applying the formula of consecutive four, we get

$\cos a \cos B = \sin a \cot c/2 - \sin B \cot \theta$ from $\triangle BCD$.

$\cos b \cos A = \sin b \cot c/2 - \sin A \cot \phi$ from $\triangle ACD$.

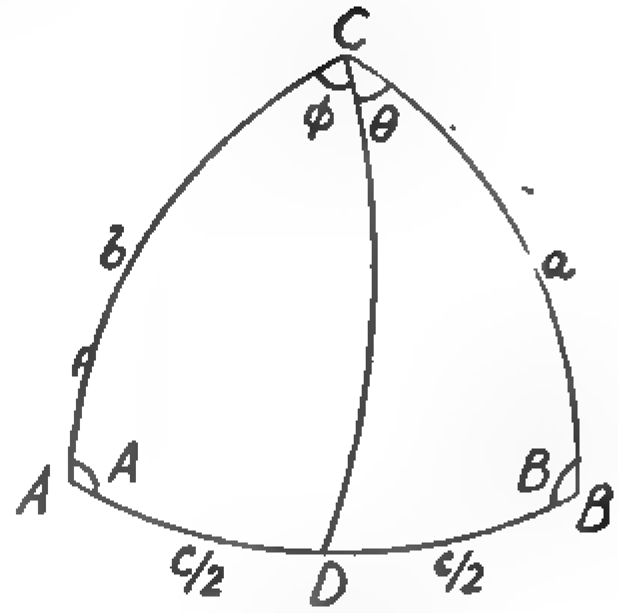


Fig. 30

$$\therefore \quad \cot\theta - \cot\phi = \cot\frac{c}{2}\left(\frac{\sin a}{\sin B} - \frac{\sin b}{\sin A}\right) - \left\{\frac{\cos a}{\sin B}\cos B - \frac{\cos b}{\sin A}\cos A\right\}.$$

Now $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C} = k$ say and $\cot\dfrac{c}{2} = \dfrac{1+\cos c}{\sin c}$;

$$\therefore \quad \cot\theta - \cot\phi = \frac{1+\cos c}{\sin c}\, k\, \frac{\sin^2 A - \sin^2 B}{\sin A \sin B}$$

$$-\left\{\frac{\cos a}{\sin B}\cdot\frac{\cos b - \cos a\cos c}{\sin a \sin c} - \frac{\cos b}{\sin A}\cdot\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right\}$$

$$= \frac{(1+\cos c)}{k \sin C}\cdot k \cdot \frac{\sin^2 A - \sin^2 B}{\sin A \sin B} - \frac{\cos c\,(\cos^2 b - \cos^2 a)}{k^2 \sin A \sin B \sin C}$$

$$= \frac{(1+\cos c)(\sin^2 A - \sin^2 B)}{\sin A \sin B \sin C} - \frac{\cos c\,(\sin^2 a - \sin^2 b)}{k^2 \sin A \sin B \sin C}.$$

Now $\sin a = k \sin A$ etc

$$= \frac{\sin^2 A - \sin^2 B}{\sin A \sin B \sin C}\,[1+\cos c - \cos c] = \frac{\sin^2 A - \sin^2 B}{\sin A \sin B \sin C}$$

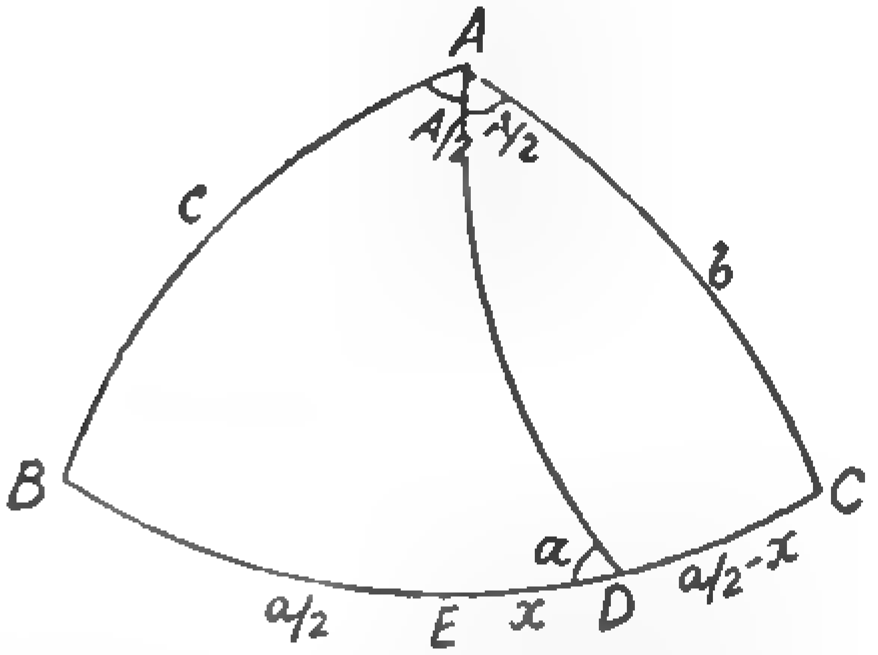
which proves 1st.

Again by sine formula we can write as

$$\frac{k^2(\sin^2 a - \sin^2 b)}{k^2 \sin a \sin b \sin C} = \frac{\sin(a+b)\sin(a-b)}{\sin a \sin b \sin C}.$$

(m) *In a spherical triangle ABC arc which bisects the angle A meets the opposite side in D, and E is the middle point of that side. Show that* $\tan x \tan\dfrac{b+c}{2} = \tan\dfrac{a}{2}\tan\dfrac{c-b}{2}$, *where* $DE = x$.

$$\frac{\sin c}{\sin \alpha}=\frac{\sin\left(\frac{a}{2}+x\right)}{\sin A/2} \qquad \text{from } ABC$$



$$\frac{\sin b}{\sin(\pi-\alpha)}=\frac{\sin\left(\frac{a}{2}-x\right)}{\sin A/2}$$

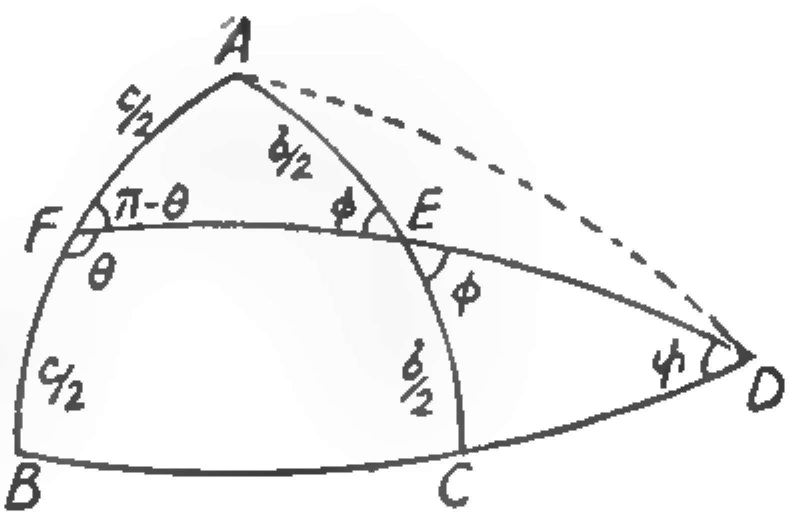
from $\triangle ACD$.

Divide and apply componendo and dividendo.

(n) *The middle points of the sides AB, AC of a spherical triangle are joined by the arc of a great circle which cuts the base produced towards O at D. Prove that $BD+CD=\pi$ and that*

$$\cos AD=\sin\frac{c+b}{2}\sin\frac{c-b}{2}\operatorname{cosec}\frac{a}{2}, \qquad \textbf{(Utkal 55)}$$

AB being the greater of the two sides.



$$\frac{\sin BD}{\sin\theta}=\frac{\sin c/2}{\sin\psi},$$

$$\frac{\sin CD}{\sin\phi}=\frac{\sin b/2}{\sin\psi}.$$

$$\therefore \quad \frac{\sin BD}{\sin CD}=\frac{\sin\theta.\sin c/2}{\sin\phi.\sin b/2}.$$

Now from $\triangle AEF$, we have

$$\frac{\sin b/2}{\sin(\pi-\theta)}=\frac{\sin c/2}{\sin\phi}; \quad \therefore \quad \frac{\sin\theta\sin c/2}{\sin\phi\sin b/2}=1.$$

Hence $\dfrac{\sin BD}{\sin CD}=1$ or $\sin BD=\sin CD$; $\therefore BD=\pi-CD$ as they cannot be equal. Therefore $BD+CD=\pi$

or $$BC+CD+CD=\pi.$$

$$\therefore \quad CD=\frac{\pi-a}{2}=\frac{\pi}{2}-\frac{a}{2}. \qquad \ldots(1)$$

Now

$$\cos AD = \cos AC \cos CD + \sin AC \sin CD \cos ACD$$
$$= \cos b \sin a/2 + \sin b \cos a/2 \cos (\pi - C) \quad \text{[by (1)]}$$

or
$$\cos AD = \cos b \sin \frac{a}{2} - \sin b \cdot \cos \frac{a}{2} \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$
$$= \cos b \sin \frac{a}{2} - \frac{\cos c - \cos a \cos b}{2 \sin a/2}$$
$$= \frac{\cos b (1 - \cos^2 a) - \cos c + \cos a \cos b}{2 \sin a/2}$$
$$= \frac{\cos b - \cos c}{2 \sin a/2} = \frac{2 \sin \frac{b+c}{2} \sin \frac{c-b}{2}}{2 \sin a/2}$$
$$= \sin \frac{c+b}{2} \sin \frac{c-b}{2} \operatorname{cosec} \frac{a}{2}.$$

4. (a) *In an equilateral triangle ABC, prove the following relations :—*

(*i*) *$2 \cos a/2 \sin A/2 = 1$.* (*ii*) *$\sec A = 1 + \sec a$.*
(Punjab 46, Delhi 56) **(Agra 53, Rajputana 51)**

(*iii*) *$1 - 2 \cos A = \tan^2 a/2$.* (*iv*) *$1 + 2 \cos a = \cot^2 A/2$.*

(*v*) *$\log \sin \frac{A}{2} + \log \cos \frac{a}{2} + \log 2 = 0$* **(Nagpur 55)**

(i)
$$2 \sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$
$$= 1 - \frac{\cos a - \cos^2 a}{\sin^2 a} \quad \because \ a = b = c$$

or
$$2 \sin^2 \frac{A}{2} = \frac{\sin^2 a + \cos^2 a - \cos a}{1 - \cos^2 a} = \frac{1 - \cos a}{(1 - \cos a)(1 + \cos a)}$$
$$= \frac{1}{1 + \cos a} = \frac{1}{2 \cos^2 \frac{a}{2}}.$$

$\therefore$ $4 \sin^2 A/2 \cos^2 a/2 = 1$ or $2 \sin A/2 \cos a/2 = 1$ which proves (i).

(ii) Writing the value of $\cos A$ and putting $a=b=c$, we get

$$\cos A=\frac{\cos a}{1+\cos a}, \quad \therefore \quad \sec A=\frac{1+\cos a}{\cos a}=\sec a+1.$$

(iii) $$1-2\cos A=1-2.\frac{\cos a}{1+\cos a}=\frac{1-\cos a}{1+\cos a}=\frac{2\sin^2 a/2}{2\cos^2 a/2}$$
$$=\tan^2 a/2.$$

Note. *From the relation $1-2\cos A=\tan^2 a/2$, deduce the limits between which the sides and the angles of an equilateral triangle are restricted.*

$\tan^2 a/2$ is clearly +ive *i. e.* >0;

$\therefore \quad 1-2\cos A>0$ or $\cos A<\frac{1}{2}$.

Hence $A>60°$.

Therefore we conclude that the angles of a spherical equilateral triangle lie between 60° and 180°.

Again the greatest value of $1-2\cos A=1-2(-1)=3$ *i. e.* greatest value of $\tan^2 a/2=3$ or of $\tan a/2$ is $\sqrt{3}$ *i. e.* $a/2=\tan^{-1}\sqrt{3}=60°$ or $a=120°$. Hence the greatest value of the sides can be 120° and least value is clearly zero showing that the sides of a spherical equilateral triangle lie between 0° and 120°.

(iv) $$\cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C}=\frac{\cos A+\cos^2 A}{\sin^2 A}$$
$$=\frac{\cos A(1+\cos A)}{(1-\cos A)(1+\cos A)}=\frac{\cos A}{1-\cos A}$$

$$\therefore \quad 1+2\cos a=1+2.\frac{\cos A}{1-\cos A}=\frac{1+\cos A}{1-\cos A}=\cot^2\frac{A}{2}$$

(v) Take log of both sides in (1) and we get the result.

(b) *If A and A′ be the angles of an equilateral triangle and its polar reciprocal, prove that $\cos A\cos A'=\cos A+\cos A'$.*

We have proved above that in an equilateral triangle,

$$\cos A=\frac{\cos a}{1+\cos a} \text{ and } A'=\pi-a; \quad \therefore \quad \cos A'=-\cos a.$$

$$\therefore \quad \cos A + \cos A' = \frac{\cos a}{1+\cos a} - \cos a = \frac{-\cos^2 a}{1+\cos a}$$
$$= \cos A . \cos A'.$$

(c) *If a be the side of an equilateral spherical triangle and a′ that of its polar triangle, show that cos a/2 cos a′/2 = ½.*

$$\cos A = \frac{\cos a}{1+\cos a}; \quad \therefore \quad \cos a = \frac{\cos A}{1-\cos A}.$$

$$\therefore \quad 2\cos^2 a/2 = 1 + \cos a = 1 + \frac{\cos A}{1-\cos A} = \frac{1}{1-\cos A}.$$

Again $a' = \pi - A \quad \therefore \quad \cos a' = -\cos A.$

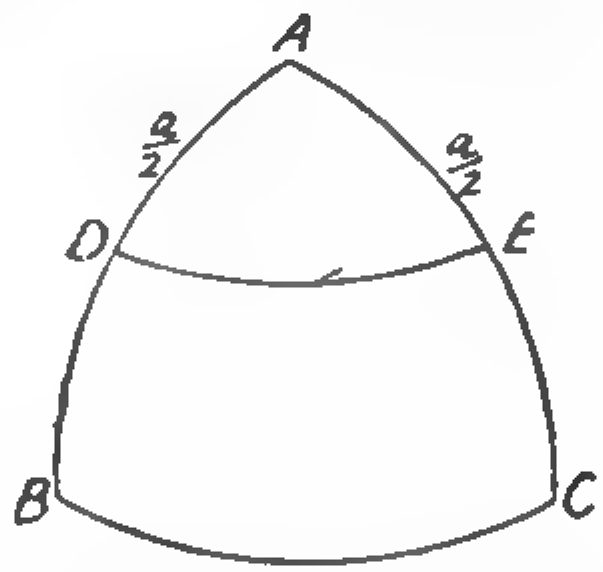
$$2\cos^2 a'/2 = 1 + \cos a' = 1 - \cos A.$$

$$\therefore \quad 2\cos^2 a/2 . 2\cos^2 a'/2 = (1-\cos A)\frac{1}{1-\cos A} = 1,$$

$$\therefore \quad 2\cos a/2 . \cos a'/2 = 1.$$

(d) *If D, E are the middle points of the sides AB, AC of an equilateral spherical triangle, prove that sin DE/2 = ½ tan a/2.*

We have proved above that in equilateral triangle



$$\cos A = \frac{\cos a}{1+\cos a}.$$

$$\cos DE = \cos a/2 . \cos a/2 + \sin a/2 . \sin a/2 . \cos A$$

$$= \cos^2 \frac{a}{2} + \sin^2 \frac{a}{2} . \frac{\cos a}{1+\cos a}$$

Fig. 33

$$\therefore \quad 2\sin^2 \frac{DE}{2} = 1 - \cos DE = 1 - \cos^2 \frac{a}{2} - \sin^2 \frac{a}{2} \frac{\cos a}{1+\cos a}$$

$$= \sin^2 \frac{a}{2} \left(\frac{1+\cos a - \cos a}{1+\cos a} \right)$$

$$= \sin^2 \frac{a}{2} . \frac{1}{2\cos^2 \frac{a}{2}}; \quad \therefore \quad \sin \frac{DE}{2} = \frac{1}{2} \tan \frac{a}{2}.$$

5. *If α, β, γ be the arcs joining the middle points of the sides of a spherical triangle ABC, show that*

$$\frac{\cos\alpha}{\cos a/2}=\frac{\cos\beta}{\cos b/2}=\frac{\cos\gamma}{\cos c/2}=\frac{1+\cos a+\cos b+\cos c}{4\cos a/2\cos b/2\cos c/2}.\quad\textbf{(Agra 48)}$$

By applying cosine formula on $\triangle AEF$, we get

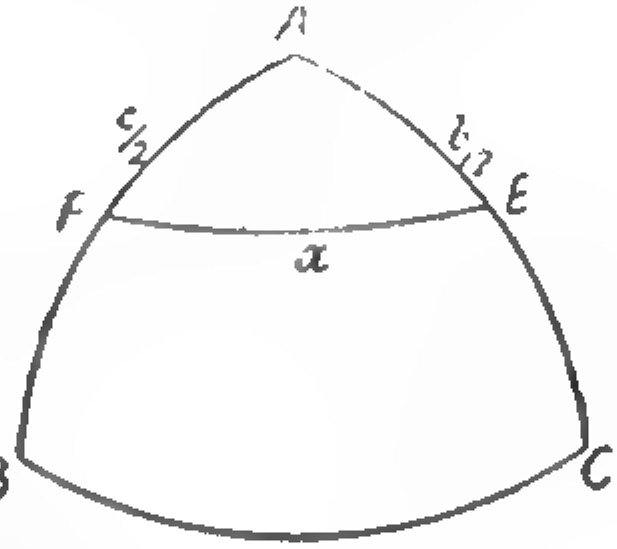


Fig. 34

$$\cos\alpha=\cos b/2\cos c/2+\sin b/2\sin c/2\cos A$$

$$=\cos b/2\cos c/2+\sin b/2\sin c/2\times\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$=\frac{4\cos^2 b/2\cos^2 c/2+(\cos a-\cos b\cos c)}{4\cos b/2\cos c/2},$$

$$\because\quad\frac{\sin b/2}{\sin b}=\frac{1}{2\cos b/2};$$

$$\therefore\quad\frac{\cos\alpha}{\cos a/2}=\frac{(1+\cos b)(1+\cos c)+\cos a-\cos b\cos c}{4\cos a/2\cos b/2\cos c/2}$$

$$=\frac{1+\cos a+\cos b+\cos c}{4\cos a/2\cos b/2\cos c/2}.$$

The symmetry of result shows that it is also equal to

$$\frac{\cos\beta}{\cos b/2}=\frac{\cos\gamma}{\cos c/2}.$$

6. *If the three sides of a spherical triangle be halved and a new triangle formed, show that the angle θ between the new sides $b/2$ and $c/2$ is given by $\cos\theta=\cos A+\frac{1}{2}\tan b/2\tan c/2\sin^2\theta$.*

(Sagar 51, 57; Nagpur 59; Agra 58; Utkal 54)

We have to prove that

$$\cos A=\cos\theta-(1/2)\tan b/2\tan c/2\sin^2\theta.$$

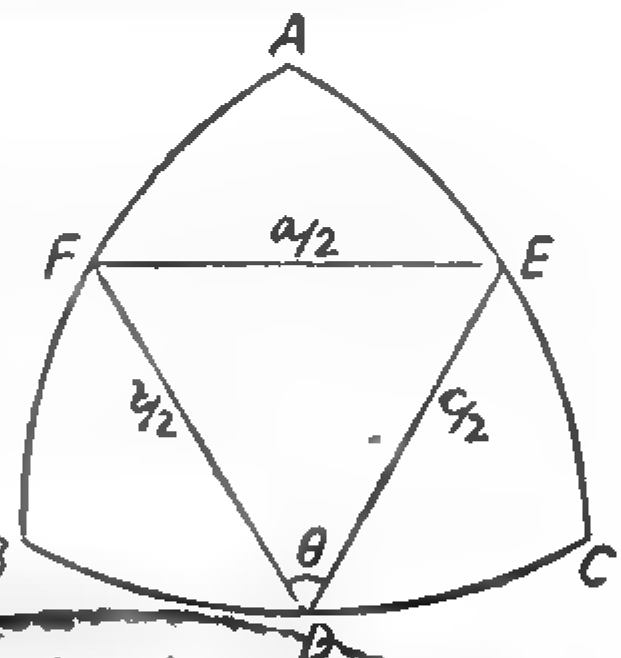


Fig. 35

Now $\cos\theta=\dfrac{\cos a/2-\cos b/2\cos c/2}{\sin b/2\sin c/2}$ from $\triangle DEF$.

$$\therefore\quad\text{R. H. S.}=\frac{\cos a/2-\cos b/2\cos c/2}{\sin b/2\sin c/2}-\frac{1}{2}\frac{\sin b/2\sin c/2}{\cos b/2\cos c/2}$$

$$+\frac{1}{2}\frac{\sin b/2 \sin c/2}{\cos b/2 \cos c/2}\cdot\left(\frac{\cos a/2-\cos b/2 \cos c/2}{\sin b/2 \sin c/2}\right)^2;$$

$$=\frac{(\cos a/2-\cos b/2\cos c/2)\,2\cos b/2\cos c/2-(\sin^2 b/2\sin^2 c/2)+(\cos^2 a/2+\cos^2 b/2\cos^2 c/2-2\cos a/2\cos b/2\cos c/2)}{2\sin b/2\sin c/2\cos b/2\cos c/2}$$

$$=\frac{-\cos^2 b/2\cos^2 c/2-\sin^2 b/2\sin^2 c/2+\cos^2 a/2}{2\sin b/2\sin c/2.\cos b/2\cos c/2}$$

$$=\frac{-\frac{1+\cos b}{2}\cdot\frac{1+\cos c}{2}-\frac{1-\cos b}{2}\cdot\frac{1-\cos c}{2}+\frac{1+\cos a}{2}}{\frac{1}{2}\sin b\sin c}$$

$$=\frac{\cos a-\cos b\cos c}{\sin b\sin c}=\cos A.$$

7. *In any spherical triangle, prove that*

$\cos a\tan B+\cos b\tan A+\tan C=\cos a\cos b\tan A\tan B\tan C$.

(Punjab 55, Agra 59, Nagpur 54, Sagar 51)

We know that $\cos a=\dfrac{\cos A+\cos B\cos C}{\sin B\sin C}$ ...(1)

$\therefore$ L. H. S.

$$=\cos a\tan B+\frac{\cos B+\cos A\cos C}{\sin A\sin C}\cdot\frac{\sin A}{\cos A}+\frac{\sin C}{\cos C}$$

$$=\cos a\frac{\sin B}{\cos B}+\frac{\cos B\cos C+\cos A\cos^2 C+\cos A\sin^2 C}{\cos A\sin C\cos C}$$

$$=\cos a\frac{\sin B}{\cos B}+\frac{\cos B\cos C+\cos A}{\cos A\sin C\cos C}$$

$$=\cos a\frac{\sin B}{\cos B}+\frac{\cos a\sin B\sin C}{\cos A\sin C\cos C}\text{ [by (1)]} \quad ...(2)$$

$$=\cos a\sin B\left[\frac{1}{\cos B}+\frac{1}{\cos A\cos C}\right]$$

$$=\cos a\sin B\left[\frac{\cos B+\cos A\cos C}{\cos A\cos B\cos C}\right]$$

$$=\cos a\sin B\frac{\cos b.\sin A\sin C}{\cos A\cos B\cos C}\text{ as above in (2)}$$

$=\cos a \cos b . \tan A \tan B \tan C.$

8. *In any spherical triangle ABC, prove that*
$\sin b \sin c+\cos b \cos c \cos A=\sin B \sin C-\cos B \cos C \cos a.$

(Rajputana 57, Utkal 55)

L. H. S.

$$=\sin b \sin c+\cos b \cos c \frac{\cos a-\cos b \cos c}{\sin b \sin c}$$

$$=\frac{(1-\cos^2 b)(1-\cos^2 c)+\cos a \cos b \cos c-\cos^2 b \cos^2 c}{\sin b \sin c}$$

$$=\frac{1-\cos^2 b-\cos^2 c+\cos a \cos b \cos c}{\sin b \sin c} \qquad \ldots(1)$$

In R. H. S. putting $\sin B=\dfrac{2n}{\sin a \sin c}$, $\sin C=\dfrac{2n}{\sin a \sin b}$

and the value of $\cos B$ and $\cos C$ in terms of sides, we get R. H. S.

$$=\frac{1-\cos^2 a-\cos^2 b-\cos^2 c+2 \cos a \cos b \cos c}{\sin a \sin c . \sin a \sin b}$$

$$-\cos a . \frac{\cos b-\cos c \cos a}{\sin c \sin a} . \frac{\cos c-\cos a \cos b}{\sin a \sin b}$$

$$=\frac{1}{\sin^2 a \sin b \sin c}\,[\sin^2 a-\cos^2 b\,(1-\cos^2 a)$$
$$-\cos^2 c\,(1-\cos^2 a)+\cos a \cos b \cos c\,(1-\cos^2 a)]$$

$$=\frac{1-\cos^2 b-\cos^2 c+\cos a \cos b \cos c}{\sin b \sin c}=\text{L. H. S.}$$

Alternative Method.

As $\cos^2 A+\sin^2 A=1$ $\therefore$ L. H. S. can be written as

$\sin b \sin c\,(\cos^2 A+\sin^2 A)+\cos b \cos c \cos A$

$=\sin b \sin c \sin^2 A+\cos A\,(\cos b \cos c+\sin b \sin c \cos A)$

$=\sin b \sin c . k^2 \sin^2 a+\cos A \cos a$ by sine formula

$\ldots \quad \ldots(1)$

Similarly since $\cos^2 a+\sin^2 a=1$,

$\therefore$ R. H. S. $=\sin B \sin C\,(\cos^2 a+\sin^2 a)-\cos B \cos C \cos a$

$$=\sin B \sin C \sin^2 a+\cos a\,(-\cos B \cos C$$
$$+\sin B \sin C \cos a)$$
$$=k \sin b . k \sin c \sin^2 a+\cos a . \cos A. \qquad \ldots(2)$$

From (1) and (2) we find that L. H. S.=R. H. S.

9. (a) *Prove that in any spherical triangle ABC,*

$$\tan b=\frac{\tan a \cos C+\tan c \cos A}{1-\tan a \tan c \cos A \cos C}.$$

(Sagar 51, Nagpur 60)

$$N^r=\frac{\sin a}{\cos a}\cdot\frac{\cos c-\cos a \cos b}{\sin a \sin b}+\frac{\sin c}{\cos c}\cdot\frac{\cos a-\cos b \cos c}{\sin b \sin c}$$

$$=\frac{1}{\sin b \cos a \cos c}\,[\cos^2 c+\cos^2 a-2 \cos a \cos b \cos c]. \qquad \ldots(1)$$

$$D^r=1-\frac{\sin a}{\cos a}\cdot\frac{\sin c}{\cos c}\cdot\frac{\cos a-\cos b \cos c}{\sin b \sin c}\cdot\frac{\cos c-\cos a \cos b}{\sin a \sin b}$$

$$=\frac{1}{\sin^2 b \cos a \cos c}\,[\cos a \cos c \sin^2 b-\{\cos a \cos c$$
$$-\cos b\,(\cos^2 c+\cos^2 a)+\cos a \cos c \cos^2 b\}]$$

$$=\frac{1}{\sin^2 b \cos a \cos c}\,[\cos b\,(\cos^2 c+\cos^2 a)$$
$$-\cos a \cos c.(1-\sin^2 b+\cos^2 b]$$

$$=\frac{1}{\sin^2 b \cos a \cos c}\,[\cos b\,(\cos^2 c+\cos^2 a)$$
$$-\cos a \cos c . 2 \cos^2 b]$$

$$=\frac{\cos b}{\sin^2 b \cos a \cos c}\,[\cos^2 c+\cos^2 a-2 \cos a \cos b \cos c] \qquad \ldots(2)$$

$$\therefore \quad \frac{N^r}{D^r}=\frac{\sin^2 b}{\cos b}\cdot\frac{1}{\sin b}=\tan b \text{ [by (1) and (2)]}.$$

Another form.

$$\tan b=\frac{\cot c \cos C+\cot a \cos A}{\cot a \cot c-\cos A \cos C}.$$

Divide above and below by cot a cot c and it is reduced to above form.

Note—See alternative method in part (o) Q 1 next exercise.

(b) *In a spherical triangle ABC if $A=a$, prove that*

$$\tan\frac{a}{2}=\frac{\tan\frac{b}{2}\sim\tan\frac{c}{2}}{1-\tan\frac{b}{2}\tan\frac{c}{2}}.$$

[Agra 54, Utkal 54, Lucknow 56, Alld. 50]

We know that $\tan^2\frac{a}{2}=\frac{1-\cos a}{1+\cos a}$.

Now $\cos a=\cos b\cos c+\sin b\sin c\cos A$. Put $A=a$

$$\therefore\quad \cos a\,(1-\sin b\sin c)=\cos b\cos c.$$

or
$$\cos a=\frac{\cos b\cos c}{1-\sin b\sin c}.$$

$$\therefore\quad \tan^2\frac{a}{2}=\frac{1-\sin b\sin c-\cos b\cos c}{1-\sin b\sin c+\cos b\cos c}$$

$$=\frac{1-\cos(b\sim c)}{1+\cos(b+c)}=\frac{2\sin^2\frac{b\sim c}{2}}{2\cos^2\frac{b+c}{2}}.$$

$$\therefore\quad \tan\frac{a}{2}=\frac{\sin\frac{b}{2}\cos\frac{c}{2}\sim\cos\frac{b}{2}\sin\frac{c}{2}}{\cos\frac{b}{2}\cos\frac{c}{2}-\sin\frac{b}{2}\sin\frac{c}{2}}.$$

Divide above and below by $\cos\frac{b}{2}\cos\frac{c}{2}$.

$$\therefore\quad \tan\frac{a}{2}=\frac{\tan\frac{b}{2}\sim\tan\frac{c}{2}}{1-\tan\frac{b}{2}\tan\frac{c}{2}}.$$

10. (a) *In any spherical triangle, prove that*

$$\cos a=\cos(b+c)+2\sin b\sin c\cos^2\frac{A}{2}.$$

Write $2\cos^2\frac{A}{2}$ as $1+\cos A$ and put the value of $\cos A$ etc.

(b) *If* $b+c=90°$, *prove that* $\cos a=\sin 2c\cos^2\frac{A}{2}$.

$$\cos a=\cos b\cos c+\sin b\sin c\cos A.$$

Now $\cos b=\cos\left(\frac{\pi}{2}-c\right)=\sin c$ and $\sin b=\cos c$.

$$\therefore\quad \cos a=\sin c\cos c\,(1+\cos A)=\sin c\cos c.2\cos^2\frac{A}{2}$$

$$=\sin 2c\cos^2\frac{A}{2}.$$

(c) *If the sum of the two sides of a spherical triangle exceeds the third side by the semi-circumference of a great circle, show that sin of half the angle contained by these sides is geometric mean between their cotangents.*

We are given $b+c=\pi+a$; $\therefore\quad \cos(b+c)=-\cos a$

or $\cos b\cos c-\sin b\sin c=-(\cos b\cos c+\sin b\sin c\cos A)$

or $2\cos b\cos c=\sin b\sin c\,(1-\cos A)$

or $\cot b\cot c=\sin^2\frac{A}{2}$, $\because\quad 1-\cos A=2\sin^2\frac{A}{2}$.

11. *Two spherical triangles ABC and A′BC standing on the same base BC are such that* $\tan B\tan C=\tan B'\tan C'$; *show that*

$$\cos A\cos B'\cos C'=\cos A'\cos B\cos C.$$

(Jabalpur 59, Sagar 52)

Equating the values of cos BC where BC is common side, we get

$$\frac{\cos A+\cos B\cos C}{\sin B\sin C}=\frac{\cos A'+\cos B'\cos C'}{\sin B'\sin C'}$$

or $$\frac{\dfrac{\cos A}{\cos B \cos C}+1}{\tan B \tan C}=\frac{\dfrac{\cos A'}{\cos B' \cos C'}+1}{\tan B' \cos C'}.$$

Now $\tan B \tan C=\tan B' \tan C'$ (given).

$$\therefore \quad \frac{\cos A}{\cos B \cos C}=\frac{\cos A'}{\cos B' \cos C'}$$

or $$\cos A \cos B' \cos C'=\cos A' \cos B \cos C.$$

12. (a) *If P is taken in AB, a side of any triangle ABC, such that AP equals AC, show that*

$$\sin c \cos CP=\cos a \sin b+\cos b \sin (c-b). \quad \textbf{(Agra 53)}$$

$$\cos CP=\cos a \cos (c-b) +\sin a \sin (c-b) \cos B$$

or $$\cos CP=\cos a \cos (c-b) +\sin a \sin (c-b) \times\frac{\cos b-\cos a \cos c}{\sin a \sin c}.$$

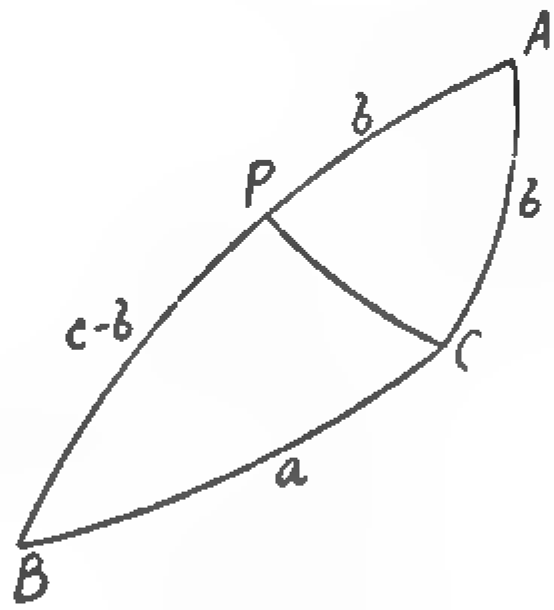


Fig. 36

$\therefore$ $\sin c \cos CP$

$$=\sin c \cos a \cos (c-b)+\sin (c-b) \cos b -\sin (c-b) \cos a \cos c$$

$$=\sin (c-b) \cos b+\cos a [\sin c \cos (c-b)-\cos c \sin (c-b)]$$

$$=\sin (c-b) \cos b+\cos a \sin \{c-(c-b)\}$$

$$=\sin (c-b) \cos b+\cos a \sin b.$$

Proved

(b) *In a spherical triangle ABC, the medians from A and B are equal. Prove that either a=b or*

$$\sin^2 \frac{c}{2}=\cos^2 \frac{a}{2}+\cos \frac{a}{2} \cos \frac{b}{2}+\cos^2 \frac{b}{2}. \quad \textbf{(Agra 56)}$$

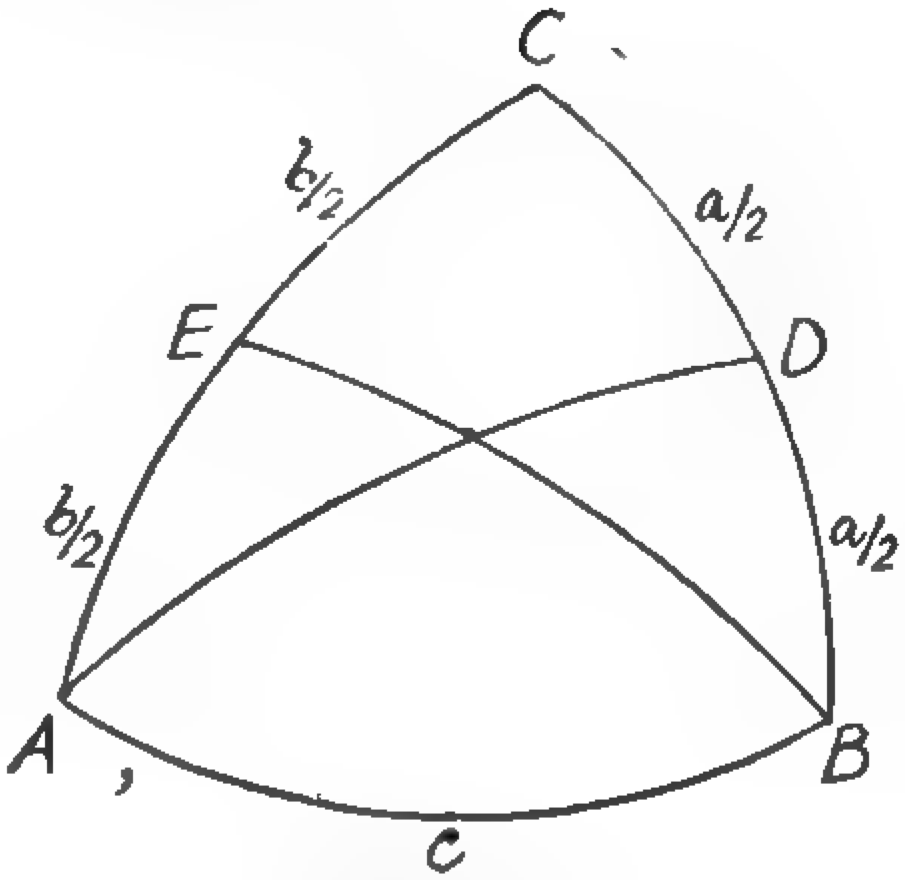


Fig. 37

Let AD and BE be the medians from A and B.

By cosine formula,

$$\cos AD = \cos\frac{a}{2}\cos c + \sin\frac{a}{2}\sin c\cos B$$

$$= \cos\frac{a}{2}\cos c + \sin\frac{a}{2}\sin c \times \frac{\cos b - \cos a\cos c}{\sin a\sin c}$$

$$= \cos\frac{a}{2}\cos c + \frac{\cos b}{2\cos\frac{a}{2}} - \frac{\cos a\cos c}{2\cos\frac{a}{2}}. \qquad \ldots(1)$$

Similarly

$$\cos BE = \cos\frac{b}{2}\cos c + \frac{\cos a}{2\cos\frac{b}{2}} - \frac{\cos b\cos c}{2\cos\frac{b}{2}} \qquad \ldots(2)$$

Since $AD = BE$ given, $\therefore$ $\cos AD = \cos BE$.

Hence from (1) and (2), we get

$$\cos c\left(\cos\frac{a}{2} - \cos\frac{b}{2}\right) + \left\{\frac{\cos b}{2\cos\frac{a}{2}} - \frac{\cos a}{2\cos\frac{b}{2}}\right\} + \left\{\frac{\cos b\cos c}{2\cos\frac{b}{2}} - \frac{\cos a\cos c}{2\cos\frac{a}{2}}\right\} = 0.$$

Put $\cos a = 2\cos^2\frac{a}{2} - 1$ and $\cos b = 2\cos^2\frac{b}{2} - 1$.

$$\cos c\left(\cos\frac{a}{2} - \cos\frac{b}{2}\right) + \frac{\cos\frac{b}{2}\left(2\cos^2\frac{b}{2} - 1\right) - \cos\frac{a}{2}\left(2\cos^2\frac{a}{2} - 1\right)}{2\cos\frac{a}{2}\cos\frac{b}{2}}$$

$$+\frac{\cos c}{2\cos\frac{a}{2}\cos\frac{b}{2}}\left\{\cos\frac{a}{2}\left(2\cos^2\frac{b}{2}-1\right)\right.$$

$$\left.-\cos\frac{b}{2}\left(2\cos^2\frac{a}{2}-1\right)\right\}=0$$

or $$\cos c\left(\cos\frac{a}{2}-\cos\frac{b}{2}\right)$$

$$+\frac{\left(\cos\frac{a}{2}-\cos\frac{b}{2}\right)-2\left(\cos^3\frac{a}{2}-\cos^3\frac{b}{2}\right)}{2\cos\frac{a}{2}\cos\frac{b}{2}}$$

$$+\frac{\cos c}{2\cos\frac{a}{2}\cos\frac{b}{2}}\left\{2\cos\frac{a}{2}\cos\frac{b}{2}\left(\cos\frac{b}{2}-\cos\frac{a}{2}\right)\right.$$

$$\left.-\left(\cos\frac{a}{2}-\cos\frac{b}{2}\right)\right\}=0$$

or $$\frac{\cos\frac{a}{2}-\cos\frac{b}{2}}{2\cos\frac{a}{2}\cos\frac{b}{2}}\left[2\cos\frac{a}{2}\cos\frac{b}{2}\cos c+1\right.$$

$$-2\left(\cos^2\frac{a}{2}+\cos\frac{a}{2}\cos\frac{b}{2}+\cos^2\frac{b}{2}\right)$$

$$\left.+\cos c\left\{-2\cos\frac{a}{2}\cos\frac{b}{2}-1\right\}\right]=0$$

$$\frac{\cos\frac{a}{2}-\cos\frac{b}{2}}{2\cos\frac{a}{2}\cos\frac{b}{2}}\left[1-\cos c-2\left(\cos^2\frac{a}{2}+\cos\frac{a}{2}\cos\frac{b}{2}\right.\right.$$

$$\left.\left.+\cos^2\frac{b}{2}\right)\right]=0$$

$\therefore$ either $\cos\frac{a}{2}-\cos\frac{b}{2}=0$, $\therefore$ $a=b$,

or $1-\cos c-2\left(\cos^2\frac{a}{2}+\cos\frac{a}{2}\cos\frac{b}{2}+\cos^2\frac{b}{2}\right)=0$

or $\sin^2\frac{c}{2}=\cos^2\frac{a}{2}+\cos\frac{a}{2}\cos\frac{b}{2}+\cos^2\frac{b}{2}$.

13. *AB, CD are quadrants on the surface of a sphere intersecting at E, the extremities being joined by great circles; show that*

$$\cos AEC=\cos AC\cos BD-\cos BC\cos AD.$$

(Allahabad 52, Benares 57, Nagpur 58, Bihar 60)

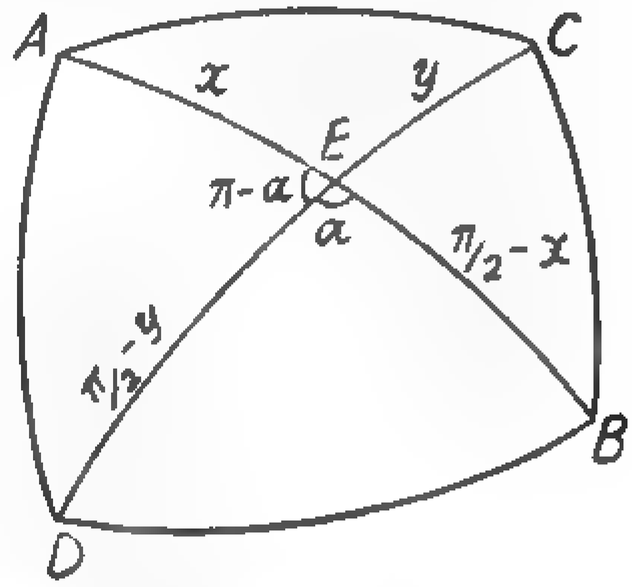
Let $AE=x$ so that

$EB=(\pi/2)-x$

and $CE=y$

so that $ED=(\pi/2)-y$.

Also let $\angle BED=\alpha$, so that



$\cos AC=\cos x\cos y+\sin x\sin y\cos\alpha$

$\cos BD=\sin x\sin y+\cos x\cos y\cos\alpha$.

Fig. 37

$$\cos AC\cos BD=(\sin x\cos x\sin y\cos y)(1+\cos^2\alpha)+\cos\alpha\{\sin^2 x\sin^2 y+\cos^2 x\cos^2 y\} \quad \ldots(1)$$

$$\cos BC=\cos y\sin x+\sin y\cos x\cos(\pi-\alpha)$$
$$\cos AD=\cos x\sin y+\sin x\cos y\cos(\pi-\alpha).$$

$\therefore$ $$\cos BC\cos AD=(\sin x\cos x\sin y\cos y)(1+\cos^2\alpha)-\cos\alpha\{\cos^2 x\sin^2 y+\sin^2 x\cos^2 y\} \quad \ldots(2)$$

$\therefore$ $\cos AC\cos BD-\cos BC\cos AD$

$=\cos\alpha\{\sin^2 y(\cos^2 x+\sin^2)+\cos^2 y(\sin^2 x+\cos^2 x)$

$=\cos\alpha(\sin^2 y+\cos^2 y)=\cos\alpha=\cos AEC.$

14. *P, Q, R, S are four points on a sphere and A is the angle between the arcs PQ and RS; show that*

$$\cos PR\cos QS-\cos PS\cos QR=\sin PQ\sin RS\cos A.$$

(Lucknow 57, Sagar 52, Nagpur 60)

$\cos PR = \cos OP \cos OR$
$\quad + \sin OP \sin OR \cos A,$

$\cos QS = \cos OQ \cos OS$
$\quad + \sin OQ \sin OS \cos A,$

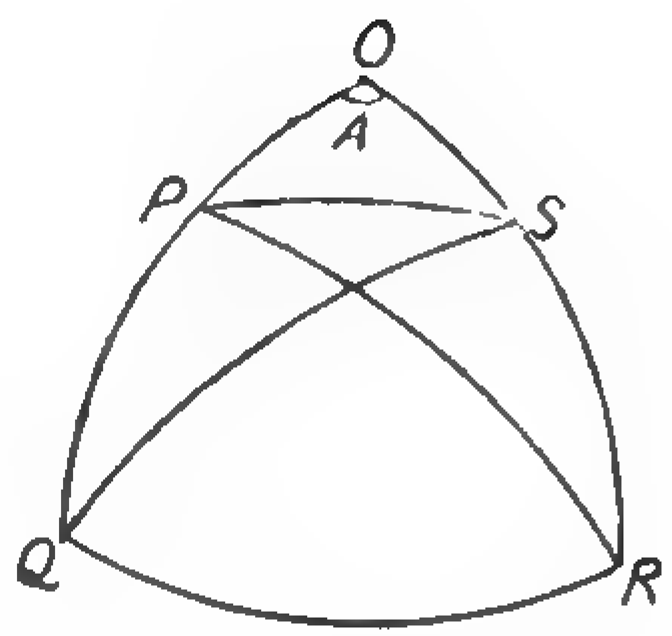


Fig. 46

$\cos PR \cos QS$

$$= (\cos OP \cos OQ \times \cos OR \cos OS) + \sin OP \sin OQ \sin OR \times \sin OS \cos^2 A$$
$$+ \cos A \{\sin OP \sin OR \cos OQ \cos OS + \sin OQ \sin OS \cos OP \sin OR\} \qquad \ldots(1)$$

$\cos PS = \cos OP \cos OS + \sin OP \sin OS \cos A$

$\cos QR = \cos OQ \cos OR + \sin OQ \sin OR \cos A.$

$\therefore \quad \cos PS \cos QR$

$$= \cos OP \cos OQ \cos OR \cos OS + \sin OP \sin OQ \sin OR \sin OS \cos^2 A$$
$$+ \cos A \{\sin OP \sin OS \cos OQ \cos OR + \sin OQ \sin OR \cos OP \cos OS\} \qquad \ldots(2)$$

Subtracting (1) and (2), we get

L.H S.

$$= \cos A [\sin OP \cos OQ (\sin OR \cos OS - \cos OR \sin OS) - \cos OP \sin OQ (\sin OR \cos OS - \cos OR \sin OS)]$$

$$= \cos A [\sin (OR - OS) \sin (OP - OQ)]$$

$$= \cos A \sin SR \sin (-PQ) = \cos A \sin RS \sin PQ.$$ Proved

15. *If a, b, c, d be the sides of a spherical quadrilateral taken in order, δ, δ′ the diagonals and φ the arc joining the middle points of the diagonals, show that*

$$\cos a + \cos b + \cos c + \cos d = 4 \cos \frac{\delta}{2} \cos \frac{\delta'}{2} \cos \phi.$$

E and F are mid. points of the diagonals AC and BD intersecting at O where $OE=x$ and $OF=y$ and $\angle EOF=\theta$ say. Since $AC=\delta$,

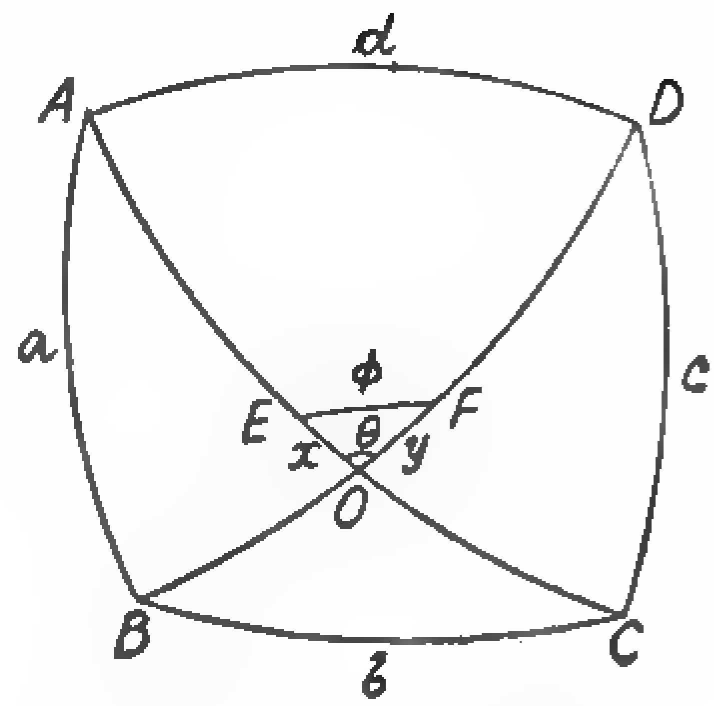


Fig. 38

$$\therefore \quad AO=AE+EO=\frac{\delta}{2}+x$$

and
$$CO=CE-OE=\frac{\delta}{2}-x.$$

Similarly,

$$BO=\frac{\delta'}{2}-y \text{ and } DO=\frac{\delta'}{2}+y.$$

$$\cos a=\cos\left(\frac{\delta}{2}+x\right)\cos\left(\frac{\delta'}{2}-y\right)$$
$$+\sin\left(\frac{\delta}{2}+x\right)\sin\left(\frac{\delta'}{2}-y\right)\cos(\pi-\theta),$$

$$\cos b=\cos\left(\frac{\delta}{2}-x\right)\cos\left(\frac{\delta'}{2}-y\right)$$
$$+\sin\left(\frac{\delta}{2}-x\right)\sin\left(\frac{\delta'}{2}-y\right)\cos\theta.$$

$$\cos a+\cos b=\cos\left(\frac{\delta'}{2}-y\right)\left[\cos\left(\frac{\delta}{2}+x\right)+\cos\left(\frac{\delta}{2}-x\right)\right]$$
$$-\sin\left(\frac{\delta'}{2}-y\right)\cos\theta\left\{\sin\left(\frac{\delta}{2}+x\right)\right.$$
$$\left.-\sin\left(\frac{\delta}{2}-x\right)\right\}$$

$$=\cos\left(\frac{\delta'}{2}-y\right).2\cos\frac{\delta}{2}\cos x$$
$$-\sin\left(\frac{\delta'}{2}-y\right)\cos\theta.2\cos\frac{\delta}{2}\sin x\ldots(1)$$

Similarly,

$$\cos c+\cos d=\cos\left(\frac{\delta'}{2}+y\right)2\cos\frac{\delta}{2}\cos x$$

$$+\sin\left(\frac{\delta'}{2}+y\right)\cos\theta . 2\cos\frac{\delta}{2}\sin x \ldots (2)$$

Adding (1) and (2), we get

$$\cos a+\cos b+\cos c+\cos d$$

$$=2\cos\frac{\delta}{2}\cos x\left[\cos\left(\frac{\delta'}{2}-y\right)+\cos\left(\frac{\delta'}{2}+y\right)\right]$$

$$+2\cos\frac{\delta}{2}\sin x\cos\theta\left[\sin\left(\frac{\delta'}{2}+y\right)-\sin\left(\frac{\delta'}{2}-y\right)\right]$$

$$=2\cos\frac{\delta}{2}\cos x . 2\cos\frac{\delta'}{2}\cos y+2\cos\frac{\delta}{2}\sin x\cos\theta$$

$$2\cos\frac{\delta'}{2}\sin y$$

$$=4\cos\frac{\delta}{2}\cos\frac{\delta'}{2}[\cos x\cos y+\sin x\sin y\cos\theta]$$

$$=4\cos\frac{\delta}{2}\cos\frac{\delta'}{2}\cos\phi \text{ from } \triangle EOF. \quad \text{Hence proved.}$$

16. *The sides AB, BC of a spherical quadrilateral are denoted by a, b respectively and the angle ABD by θ, show that*

$$\tan\theta=\frac{\cos a\sin b-\sin a(\cos b\cos B+\cot C\sin B)}{-\cot A\sin b+\sin a(\cos b\sin B-\cot C\cos B)}.$$

(Sagar 50)

Applying formula of consecutive four on $\triangle ABD$ and CBD,

$$\cos a\cos\theta=\sin a\cot x-\sin\theta\cot A \ldots (1)$$

and $\cos b\cos(B-\theta)$

$$=\sin b\cot x-\sin(B-\theta)\cot C \ldots (2$$

In order to eliminate x multiply (1) by $\sin b$ and (2) by $\sin a$ and subtract.

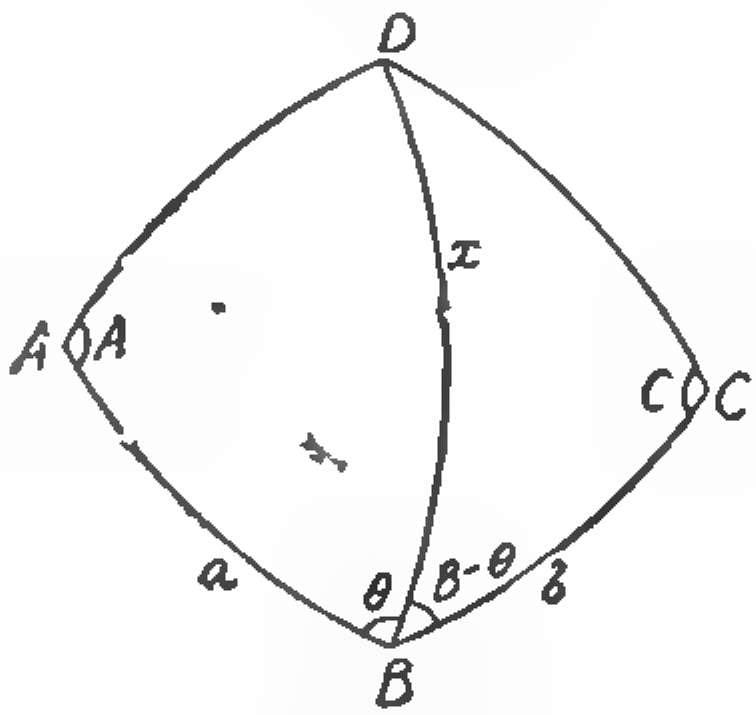


Fig. 29

$$\therefore \quad \sin b\cos a\cos\theta-\sin a\cos b\cos(B-\theta)$$

$$=\sin a\sin(B-\theta)\cot C-\sin b\sin\theta\cot A.$$

Collect the terms of $\cos\theta$ on one side and $\sin\theta$ on the other.

$$\therefore \cos\theta\,[\sin b\cos a-\sin a\cos b\cos B-\sin a\sin B\cot C]$$
$$=\sin\theta\,[\sin a\cos b\sin B-\sin a\cos B\cot C-\sin b\cot A]$$

$\therefore$ $\tan\theta=$etc. as given.

17. *The sides of a spherical triangle are all quadrants and α, β, γ are the arcs joining any point within the triangle to the angular points, prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.*

Since $AB=AC=90^\circ$,

$\therefore$ A is pole of BC

i e. $\angle BAC=90^\circ$.

Join P to to the vertices and let $PA=\alpha$, $PB=\beta$ and $PC=\gamma$.

Suppose that $\angle BAP=\theta$,

so that $\angle CAP=90-\theta$,

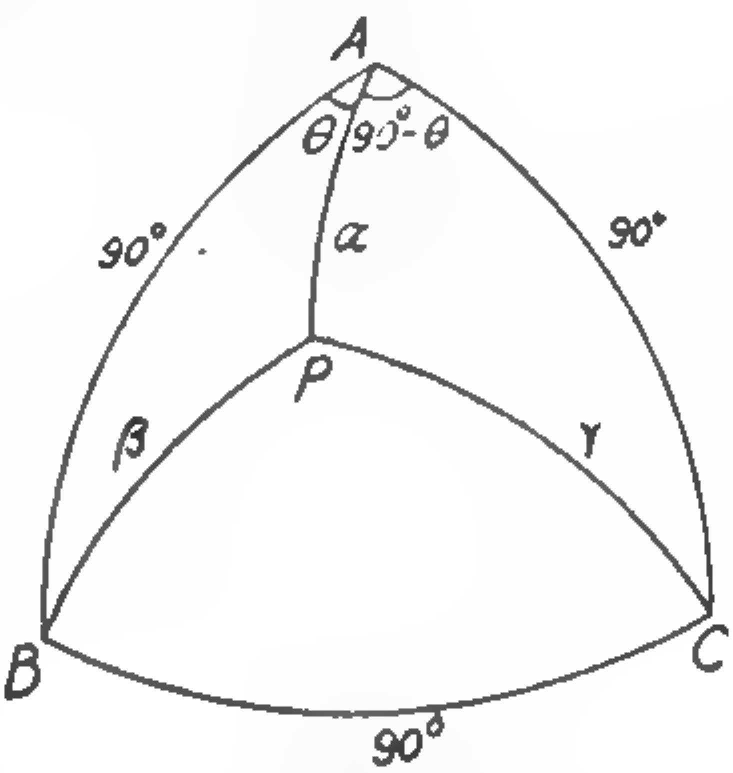


Fig. 30

$$\cos\beta=\cos\alpha\cos 90+\sin\alpha\sin 90\cos\theta \quad \text{from } \triangle ABP$$
$$=\sin\alpha\cos\theta$$

$$\cos\gamma=\cos\alpha\cos 90+\sin\alpha\sin 90\cos(90-\theta) \text{ from } \triangle APC$$
$$=\sin\alpha\sin\theta.$$

$$\therefore\quad \cos^2\beta+\cos^2\gamma=\sin^2\alpha\,(\cos^2\theta+\sin^2\theta)=1-\cos^2\alpha.$$
$$\therefore\quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

18. *In any spherical triangle if $b+c=60^\circ$, show that*

$$\cos(b-c)=\cos a+\frac{\cos\frac{3}{2}a}{2\cos(a/2)}\tan^2\frac{A}{2}.$$

$$\cos a=\cos b\cos c+\sin b\sin c\cos A \qquad \ldots(1)$$
$$=\cos b\cos c+\sin b\sin c\left(1-2\sin^2\frac{A}{2}\right)$$

or
$$\cos(b-c)=\cos a+2\sin b\sin c\sin^2\frac{A}{2}. \qquad \ldots(2)$$

Now we have to eliminate $\sin b\sin c$ in R. H. S. of (2).

$\cos(b+c)=\cos b\cos c-\sin b\sin c=\frac{1}{2}$, $\because$ $b+c=60^\circ$

$$\therefore \quad \cos b \cos c = \tfrac{1}{2} + \sin b \sin c.$$

Putting in (1), we get

$$\cos a = \tfrac{1}{2} + \sin b \sin c + \sin b \sin c \cos A$$

or $$\frac{2\cos a - 1}{2} = \sin b \sin c\,(1+\cos A) = \sin b \sin c \,.\, 2\cos^2\frac{A}{2}.$$

Putting for $\sin b \sin c$ in (2), we get

$$\cos(b-c) = \cos a + 2\sin^2\frac{A}{2} \cdot \frac{2\cos a - 1}{4\cos^2(A/2)},$$

$$= \cos a + \tfrac{1}{2}\tan^2(A/2).(2\cos a - 1)$$

$$= \cos a + \frac{1}{2}\tan^2\frac{A}{2} \cdot \frac{2\cos a\cos(a/2) - \cos(a/2)}{\cos(a/2)}$$

$$= \cos a + \frac{1}{2}\tan^2\frac{A}{2} \cdot \frac{\cos(3a/2) + \cos(a/2) - \cos(a/2)}{\cos(a/2)}$$

or $$\cos(b-c) = \cos a + \frac{\cos\frac{3}{2}a}{2\cos(a/2)}\tan^2\frac{A}{2}.$$ Hence proved.

19. *Prove that in a spherical triangle ABC in which the angle A is small, $B+C=\pi - A\cos b$, approximately.*

(Agra 1947, Raj. 58)

We have, by supplemental cosine formula,

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b.$$

We are given that A is small; $\therefore$ $\cos A = 1$ and $\sin A = A$ approximately.

$$\cos B = -\cos C + A\sin C\cos b = \cos(\pi - C) + A\sin C\cos b;$$

$$\therefore \quad B = \cos^{-1}[\cos(\pi - C) + A\sin C\cos b] = f(A) \text{ say}$$

$$= f(0) + Af'(0) + \ldots \qquad \because \ A \text{ is small.}$$

Now $$f(0) = \cos^{-1}\cos(\pi - C) = \pi - C.$$

Again $$f'(A) = -\frac{\sin C\cos b}{\sqrt{[1 - \{\cos(\pi - C) + A\sin C\cos b\}^2]}},$$

$$\therefore \quad f'(0) = -\frac{\sin C\cos b}{\sqrt{(1-\cos^2 C)}} = -\cos b;$$

$$\therefore \quad B = \pi - C + A(-\cos b).$$

$$\therefore \quad B + C = \pi - A\cos b.$$

Alternative Method.

Since A is small, we have proved that

$$\cos B = -\cos C + A \sin C \cos b. \qquad \ldots(1)$$

To a first approximation we can say that

$$\cos B = -\cos C = \cos(\pi - C);$$

$$\therefore \quad B = \pi - C.$$

Hence $B = \pi - C + x$ where x is small.

Putting in (1), we get

$$\cos(\pi - C + x) = -\cos C + A \sin C \cos b$$

or $$-\cos(C - x) = -\cos C + A \sin C \cos b$$

or $$-\{\cos C \cos x + \sin C \sin x\} = -\cos C + A \sin C \cos b.$$

Since x is small $\cos x = 1$ and $\sin x = x$;

$$\therefore \quad -\cos C - \sin C . x = -\cos C + A \sin C \cos b.$$

$$\therefore \quad x = -A \cos b.$$

Hence $$B = \pi - C + x = \pi - C - A \cos b$$

or $$B + C = \pi - A \cos b.$$ Proved.

20. *If c_1, c_2 be the two values of the third side when A, a, b are given and the triangle is ambiguous, show that*

$$\tan \frac{c_1}{2} \tan \frac{c_2}{2} = \tan \frac{b-a}{2} \tan \frac{b+a}{2}.$$

(Gorakhpur 60, Utkal 59, 56, 54, Luck. 1951)

We know that

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

or $$\cos a = \cos b \frac{1 - \tan^2 (c/2)}{1 + \tan^2 (c/2)} + \sin b \cos A \frac{2 \tan (c/2)}{1 + \tan^2 (c/2)}.$$

Arranging as a quadratic in $\tan \frac{c}{2}$, we get

$$(\cos a + \cos b) \tan^2 \frac{c}{2} - 2 \sin b \cos A \tan \frac{c}{2} + (\cos a - \cos b) = 0.$$

Since c_1, c_2 are two values of c, its roots are

$$\tan \frac{c_1}{2}, \tan \frac{c_2}{2}.$$

$$\therefore \quad \tan\frac{c_1}{2}\tan\frac{c_2}{2}=\frac{\cos a-\cos b}{\cos a+\cos b} \qquad \text{[Product of roots]}$$

$$=\frac{2\sin\frac{a+b}{2}.\sin\frac{b-a}{2}}{2\cos\frac{a+b}{2}\cos\frac{a-b}{2}}=\tan\frac{b-a}{2}\tan\frac{b+a}{2}.$$

21. (a) *A spherical triangle ABC has the angle C = 120° and b = 3a. Show that the length of arc d bisecting C and terminated by the opposite side is given by the equation*

$$\tan d=\tfrac{1}{4}(\tan a+\tan 2a).$$

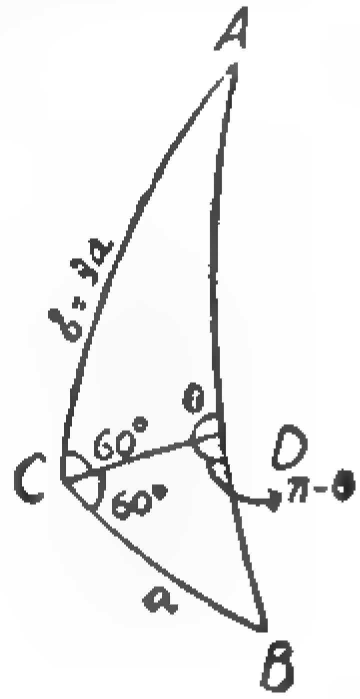


Fig. 43

Applying the formula of consecutive four on triangles ACD and BCD, we get

$$\cos d.\cos 60=\sin d\cot 3a - \sin 60\cot\theta \; [\triangle ACD]$$

$$\cos d.\cos 60=\sin d\cot a - \sin 60\cot(\pi-\theta). \; [\triangle BCD]$$

Adding there by eliminating θ, we get

$$2\cos d.\tfrac{1}{2}=\sin d(\cot 3a+\cot a).$$

$$\therefore \quad \cot d=\frac{\cos 3a\sin a+\sin 3a\cos a}{\sin 3a\sin a}$$

$$=\frac{\sin 4a}{\sin a\sin(a+2a)}.$$

$$\therefore \quad \tan d=\frac{\sin a(\sin a\cos 2a+\cos a\sin 2a)}{2\cos 2a(2\sin a\cos a)}$$

$$=\tfrac{1}{4}(\tan a+\tan 2a).$$

(b) *If the middle point of AB be equidistant from the vertices A, B, C of a spherical triangle, prove that* $\cot A\cot B=\cos^2 c/2$.

We are given that

$$DC=DB=DC=c/2,$$

$\because$ D is mid. point of AB.

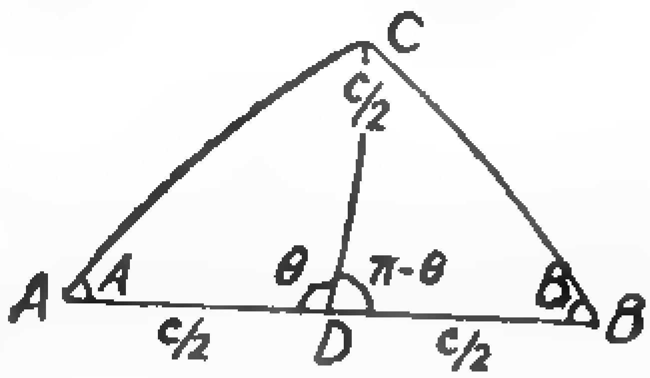


Fig. 44

Applying the formula of consecutive four on $\angle CDA$ and CDB,

we get

$$\cos(c/2)\cos\theta=\sin(c/2)\cot(c/2)-\sin\theta\cot A.$$

or $$\cos(c/2)(1-\cos\theta)=\sin\theta\cot A \text{ from } ACD. \quad \ldots(1)$$

Replacing θ by $\pi-\theta$ and A by B, we get from $\triangle BCD$,

$$\cos(c/2)(1+\cos\theta)=\sin\theta\cot B. \quad \ldots(2)$$

Multiplying the results, (1) and (2) we get

$$\cos^2(c/2)\ 1-\cos^2\theta)=\sin^2\theta.\cot A\cot B$$

or $$\cos^2(c/2)=\cot A\cot B.$$

(c) *In a spherical triangle ABC if the angle B is equal to side c, show that*

$$\sin(A-a)=\sin a\sin A\cos B\cot B.$$

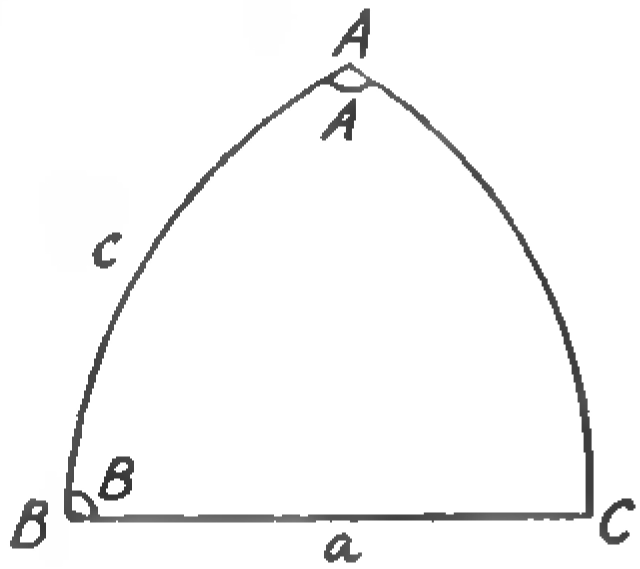


Fig. 45

$$\sin(A-a)=\sin A\cos a-\cos A\sin a$$
$$=\sin a\sin A(\cot a-\cot A)\ldots(1)$$

Applying the formula of consecutive four on $\triangle ABC$, we get

$$\cos c\cos B=\sin c\cot a-\sin B\cot A.$$

Now $c=B$, $\therefore$ $\sin c=\sin B$.

Dividing by $\sin B$ and replacing c by B, we get

$$\cos B\cot B=\cot a-\cot A.$$

Hence from (1), we get

$$\sin(A-a)=\sin a\sin A\cos B\cot B.$$ **Hence proved.**

22. *If a spherical triangle be equal and similar to its polar triangle, show that*

(*a*) $\sec^2 A+\sec^2 B+\sec^2 C+2\sec A\sec B\sec C=1.$

and deduce that such a triangle cannot be equilateral.

(Rajputana 56, '8, Agra 55, Vikram 59)

(*b*) $\sec^2 a+\sec^2 b+\sec^2 c-2\sec a\sec b\sec c=1.$

We are given that $A=A'$ and $\therefore$ $=\pi-a$ etc.

and $a=a'$ and $\therefore$ $=\pi-A$ etc.

Now $\cos A=-\cos B\cos C+\sin B\sin C\cos a.$

Put $a=\pi-A$.

$\therefore\quad \cos A=-\cos B\cos C-\sin B\sin C\cos A.$

Divide by $\cos A\cos B\cos C$.

$$\therefore\quad \sec B\sec C+\sec A=-\tan B\tan C.$$

Squaring, we get

$$\sec^2 B\sec^2 C+\sec^2 A+2\sec A\sec B\sec C=(\sec^2 B-1)(\sec^2 C-1).$$

$\therefore\quad \sec^2 A+\sec^2 B+\sec^2 C+2\sec A\sec B\sec C=1,$

which proves (a).

In case the triangle be equilateral then $A=B=C$.

$\therefore\quad 3\sec^2 A+2\sec^3 A=1 \quad\text{or}\quad \cos^3 A-3\cos A-2=0$

or $\quad(\cos A-2)(\cos^2 A+2\cos A+1)=0$

or $\quad(\cos A-2)(\cos A+1)^2=0$

Now $\cos A$ cannot b 2 and hence we get $\cos A=-1$,

$$\therefore\quad A=\pi=B=C.$$

$\therefore\quad A+B+C=3\pi$ i.e. 6 right angles, but it is also not possible as sum of the angles of a spherical triangle is less than six right angles. (§ 22 c P. 13)

Similarly starting with cosine formula, we can prove part (b).

23. (a) *In a spherical triangle ABC, $a+b+c=180°$, ; prove that*

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1.$$

$$\sin^2\frac{A}{2}=\frac{\sin(s-b)\sin(s-c)}{\sin b\sin a}=\frac{\cos b\cos c}{\sin b\sin c}=\cot b\cot c.$$

$\because\quad 2s=a+b+c=180°, \qquad \therefore\quad s=90°.$

$\therefore\quad \Sigma\sin^2(A/2)=\cot b\cot c+\cot c\cot a+\cot a\cot b.$

Now $\tan(a+b+c)$

$$=\frac{\tan a+\tan b+\tan c-\tan a\tan b\tan c}{1-\tan a\tan b-\tan b\tan c-\tan c\tan a}=0,$$

$\because\quad a+b+c=180°$ and $\tan 180°=0$.

$\therefore\quad \tan a+\tan b+\tan c=\tan a\tan b\tan c,$

Dividing by $\tan a \tan b \tan c$.

$$\cot b \cot c+\cot c \cot a+\cot a \cot b=1.$$

(b) *If the angles of a triangle be together equal to four right angles, prove that*

$$\cos^2 \frac{a}{2}+\cos^2 \frac{b}{2}+\cos^2 \frac{c}{2}=1. \qquad \textbf{(Agra 48, 54)}$$

$$\cos^2 \frac{a}{2}=\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}.$$

Now $2S=A+B+C=2\pi$, $\therefore$ $S=\pi$.

$$\therefore \quad \cos^2 \frac{a}{2}=\frac{(-\cos B)(-\cos C)}{\sin B \sin C}=\cot B \cot C.$$

Now proceeding as above in part (a), we prove the result.

(c) *In a spherical triangle if $\cos C=-\tan (a/2) \tan (b/2)$; then $C=A+B$.* **(Agra 44, Rajputana 60)**

$$-\tan \frac{a}{2} \tan \frac{b}{2}=-\sqrt{\left\{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}\right\}} \times \sqrt{\left\{\frac{-\cos S \cos (S-B)}{\cos (S-A) \cos (S-C)}\right\}}$$

$$=\frac{\cos S}{\cos (S-C)}=\cos C \text{ (given)}$$

or $\cos S=\cos (S-C) \cos C$

or $2 \cos S=2 \cos (S-C) \cos C$

or $2 \cos S=\cos S+\cos (S-2C)$ or $\cos S=\cos (S-2C)$.

$$\therefore \quad S=\pm(S-2C).$$

If we take +ive sign, we get $C=0$ which is not possible.

Hence taking −ve sign, we get

$$S=-S+2C \quad \text{or} \quad 2S=2C$$

or $A+B+C=2C$ or $C=A+B$.

(d) *In any spherical triangle, show that*

$$\sin \frac{a+b+c}{2} \sin \frac{A}{2}=\cos \frac{B}{2} \cos \frac{C}{2} \sin a.$$

$$\frac{\cos(B/2).\cos(C/2)}{\sin(A/2)}=\left[\left\{\frac{\sin s\sin(s-b)}{\sin a\sin c}\cdot\frac{\sin s\sin(s-c)}{\sin a\sin b}\right\}\right.$$
$$\left.\times\left\{\frac{\sin b\sin c}{\sin(s-b)\sin(s-c)}\right\}\right]^{1/2}$$

$$=\frac{\sin s}{\sin a}=\frac{\sin\left(\frac{a+b+c}{2}\right)}{\sin a}.$$

24. *If the sides a, b, c of a spherical triangle ABC are in arithmetical progression, show that*

$$\cos\frac{C-A}{2}=\frac{\sin^2(B/2}{\sin(A/2)\sin(C/2)}.$$

Now we know by De Alembert's Analogy (§ 10 P. 30) that

$$\frac{\cos\frac{C-A}{2}}{\sin(B/2)}=\frac{\sin\frac{c+a}{2}}{\sin(b/2)} \qquad \ldots(1)$$

and
$$\frac{\sin(B/2)}{\sin(A/2)\sin(C/2)}=\frac{\sin b}{\sin(s-b)} \qquad \ldots(2)$$

[on putting the values of sin $(A/2)$ etc.]

Now a, b, c are in A. P.

$$\left.\begin{array}{l}\therefore\quad b=\dfrac{a+c}{2}\text{ and }2s=a+b+c=3b\\ \text{or}\quad 2s-2b=b\quad\text{or}\quad s-b=b/2.\end{array}\right\}\ldots A$$

Hence from (1) and (2) by the help of A we get that R. H. S. of (1) = R. H. S. of (2); therefore L. H. S.'s are also equal and we get the result.

25. *If δ be the length of the arc from the vertex of an isosceles triangle dividing the base into segments α and β, then prove that*

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\tan\frac{a+\delta}{2}\tan\frac{a-\delta}{2}.$$

We know the following Napier's Analogies :—

$$\tan\frac{a+b}{2}=\frac{\cos\dfrac{A-B}{2}}{\cos\dfrac{A+B}{2}}\tan\frac{c}{2},$$

$$\tan\frac{a-b}{2}=\frac{\sin\dfrac{A-B}{2}}{\sin\dfrac{A+B}{2}}\tan\frac{c}{2}.$$

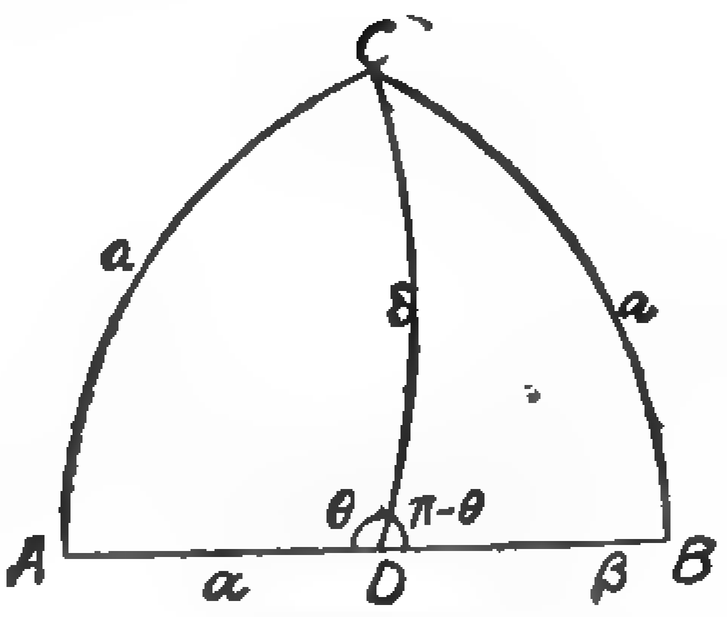


Fig. 46

Applying the above analogies on Δs ACD and BCD, we get

$$\tan\frac{a+\delta}{2}=\frac{\cos\dfrac{\theta-A}{2}}{\cos\dfrac{\theta+A}{2}}\tan\frac{\alpha}{2}. \qquad \ldots(1)$$

$\because$ $a=b$ (given), $\therefore$ $\angle A=\angle B$ by sine formula.

$$\tan\frac{a-\delta}{2}=\frac{\sin\dfrac{\pi-\theta-A}{2}}{\sin\dfrac{\pi-\theta+A}{2}}\tan\frac{\beta}{2}=\frac{\cos\dfrac{\theta+A}{2}}{\cos\dfrac{\theta-A}{2}}\tan\frac{\beta}{2}.\,.(2)$$

Multiplying 1 and 2 we get the required result.

(b) *If D be any point in the base AB of an isosceles spherical triangle ABC, prove that*

$$\frac{\cos\frac{1}{2}(AD-DB)}{\cos\dfrac{AB}{2}}=\frac{\cos CD}{\cos CA}.$$

With reference to the figure of part (a), we have to prove that

$$\frac{\cos\dfrac{\alpha-\beta}{2}}{\mathrm{c}\ \dfrac{\alpha+\beta}{2}}=\frac{\cos\delta}{\cos a}. \quad \because \quad CA=CB=a$$

Applying componendo and dividendo and simplifying. we get the same as part (a).

26. *In a spherical triangle if $A=B=2C$, show that*

(a) $8 \sin \{a+(c/2)\} \sin^2 (c/2) \cos (c/2)=\sin^3 a$

(Allahabad 58, Utkal 50, 55, Agra 1950)

(b) $8 \sin^2 (C/2) \{\cos s+\sin (C/2) \cos (c/2)=\cos a.$

(Sagar 1952)

(c) $\cos a \cos (a/2)=\cos \{c+(a/2)\}.$ **(Utkal 54)**

(a) We know that $\dfrac{\sin A}{\sin a}=\dfrac{\sin B}{\sin b}=\dfrac{\sin C}{\sin c}$.

Since $A=B$, $\therefore$ $\sin a=\sin b$ and $\cos a=\cos b$. ...(1)

Again $A=B=2C$. $\therefore$ $\dfrac{\sin 2C}{\sin a}=\dfrac{\sin C}{\sin c}$.

$\therefore$ $\sin a=2 \sin c \cos C=\sin b$. ...(2)

$$\text{L.H.S.}=8\left(\sin a \cos \frac{c}{2}+\cos a \sin \frac{c}{2}\right) \sin^2 \frac{c}{2} \cos \frac{c}{2}$$

$$=4 \sin \frac{c}{2} \cos \frac{c}{2}$$

$$\times\left\{2 \sin a \cos \frac{c}{2} \sin \frac{c}{2}+2 \cos a \sin^2 \frac{c}{2}\right\}$$

$$=2 \sin c \{\sin a \sin c+\cos a (1-\cos c)\}$$

$$=2 \sin c \{\sin a \sin c+\cos a-\cos a \cos c\}. \quad ...(3)$$

Now $\cos b=\cos a \cos c+\sin a \sin c \cos B$

$\therefore$ $\cos a \cos c=\cos b-\sin a \sin c \cos 2C$ $\because$ $B=2C$.

Putting for $\cos a \cos c$ in (3), we get

$$\text{L.H.S.}=2 \sin c \{\sin a \sin c+\cos a-\cos b +\sin a \sin c \cos 2C\}$$

$$=2 \sin c \sin a \sin c \{1+\cos 2C\}$$

[$\because$ $\cos a=\cos b$ by (1)]

$$=2 \sin a \sin^2 c . 2 \cos^2 C$$

$$=\sin a (2 \sin c \cos C)^2$$

$$=\sin a \sin^2 a \text{ [by (2)]}=\sin^3 a.$$

Alternative Method.

Since $A=B$, $\therefore$ $a=b$.

$$\therefore \quad s=\tfrac{1}{2}(a+b+c)=a+\frac{c}{2} \quad \text{or} \quad b+\frac{c}{2}. \qquad \ldots(1)$$

Again $A=B=2C$ $\therefore$ $C=(A/2)$ or $\sin C=\sin (A/2)$

or $$2 \sin (C/2) \cos (C/2)=\sin (A/2). \qquad \ldots(2)$$

Substituting the values of $\sin (C/2)$ etc. in terms of sides in (2).

$$2.\left\{\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b} \cdot \frac{\sin s \sin (s-c)}{\sin a \sin b}\right\}^{1/2}$$
$$=\left\{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}\right\}^{1/2}$$

Cancel $\sin (s-b) \sin (s-c)$ from both sides and put down the values of $s=a+(c/2)$ and $s-a=(c/2)$ from (1).

$$\therefore \quad \frac{2}{\sin^2 a}[\sin (c/2) \sin \{a+(c/2)\}]^{1/2}$$
$$=\frac{1}{[\sin a \,.\, 2 \sin (c/2) \cos (c/2)]^{1/2}}. \qquad \because \quad a=b$$

Squaring and cross-multiplying, we get

$$8 \sin^2 (C/2) \sin \{a+(c/2)\} \cos (c/2)=\sin^3 a.$$

(b) We have to prove that

$$8 \sin^2 (C/2) \ \{\cos s+\sin (C/2)\} \cos (c/2)=\cos a.$$

Just as in last part, $2 \sin (C/2) \cos (C/2)=\sin (A/2)$ [by (2)]

or $$4 \sin^2 (C/2) \cos^2 (C/2)=\sin^2 (A/2)$$

In the relation to be proved we require $\sin (C/2)$ and as such we put the values of $\cos (C/2)$ and $\sin (A/2)$.

$$\therefore \quad 4 \sin^2 (C/2).\frac{\sin s \sin (s-c)}{\sin a \sin b}=\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}$$

or $$4 \sin^2 (C/2) \sin s.\sin c=\sin a.\sin (s-b)$$

or $$4 \sin^2 (C/2) \sin s . 2 \sin (c/2) \cos (c/2)=\sin a.\sin (c/2)$$

$$\because \quad s-b=(c/2). \qquad \text{[by (1) of part (a)]}$$

or $$8 \sin^2 (C/2) \sin s.\cos (c/2)=\sin a. \qquad \ldots(1)$$

Now $\cos s+\sin (C/2)=\cos s+\sqrt{\left(\dfrac{\sin (s-a) \sin (s-b)}{\sin a \sin b}\right)}$

$$=\cos s+\frac{\sin (s-a)}{\sin a} \quad \because \quad a=b$$

$$=\frac{\cos s \sin a+\sin s \cos a-\cos s \sin a}{\sin a}=\frac{\sin s \cos a}{\sin a}.$$

$$\therefore \quad \sin s=\{\cos s+\sin (C/2)\} \frac{\sin a}{\cos a} \qquad \ldots(2)$$

Putting the value of $\sin s$ from (2) in (1), we get

$$8 \sin^2 (C/2) \{\cos s+\sin (C/2)\}.\frac{\sin a}{\cos a} \cos (c/2)=\sin a.$$

$$8 \sin^2 \frac{C}{2} \left\{\cos s+\sin \frac{C}{2}\right\} \cos \frac{c}{2}=\cos a.$$ Hence proved.

27. *Show that cosine of the angle between the chords of two sides b, c of a spherical triangle is equal to*

$$\sin \frac{b}{2} \sin \frac{c}{2}+\cos \frac{b}{2} \cos \frac{c}{2} \cos A.$$

Let O be the centre of the sphere whose radius is r and ABC be the spherical triangle whose chords are denoted by dotted lines and we have to find the angle between the chords AB and AC. If this angle be θ, then from plane triangle ABC,

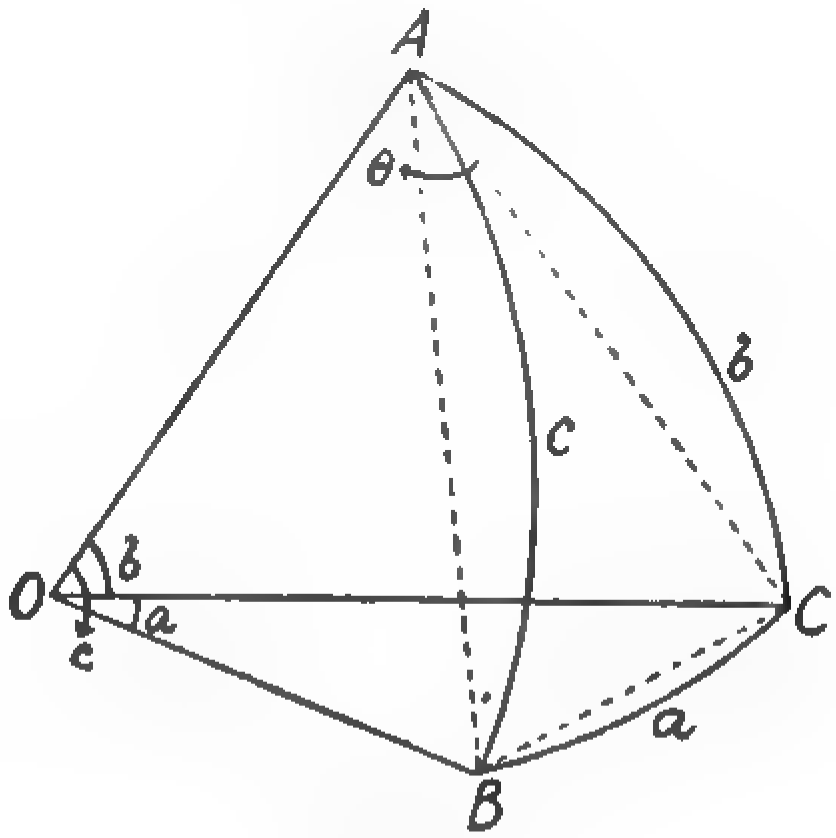


Fig. 47.

$$\cos \theta=\frac{AB^2+AC^2-BC^2}{2AB.AC}. \qquad \ldots(1)$$

Again from the plane triangle OBC, we have

$$BC^2=OB^2+OC^2-2OB.OC\cos BOC$$
$$=2r^2(1-\cos a)$$

or $BC^2=2r^2[1-(\cos b\cos c+\sin b\sin c\cos A)]$

$$=2r^2\left[1-\left(1-2\sin^2\frac{b}{2}\right)\left(1-2\sin^2\frac{c}{2}\right)-2\sin\frac{b}{2}\cos\frac{b}{2}.2\sin\frac{c}{2}\cos\frac{c}{2}\cos A\right]$$

$$=2r^2\left[2\sin^2\frac{b}{2}+2\sin^2\frac{c}{2}-4\sin^2\frac{b}{2}\sin^2\frac{c}{2}-4\sin\frac{b}{2}\sin\frac{c}{2}\cos\frac{b}{2}\cos\frac{c}{2}\cos A\right]. \quad \ldots(2)$$

Again $CA^2=2r^2(1-\cos b)=4r^2\sin^2\frac{b}{2}$ $\quad\ldots(3)$

and $\quad AB^2=2r^2(1-\cos c)=4r^2\sin^2\frac{c}{2}.$ $\quad\ldots(4)$

Putting the value of BC^2, CA^2 and AB^2 from (2), (3) and (4) in (1),

$$\cos\theta=4r^2\frac{\left[\sin^2\frac{b}{2}+\sin^2\frac{c}{2}-\sin^2\frac{b}{2}-\sin^2\frac{c}{2}+2\sin^2\frac{b}{2}\sin^2\frac{c}{2}+2\sin\frac{b}{2}\sin\frac{c}{2}\cos\frac{b}{2}\cos\frac{c}{2}\cos A\right]}{2.2r\sin\frac{b}{2}.2r.\sin\frac{c}{2}}$$

$$=\sin\frac{b}{2}\sin\frac{c}{2}+\cos\frac{b}{2}\cos\frac{c}{2}\cos A.$$ Hence proved.

28. *If DE be an arc of a great circle bisecting the sides AB, AC of a spherical triangle at D and E and P is pole of DE. PB, PD, PE, PC be joined by arcs of great circles, show that angle BCP=twice the angle DPE.*

(Nagpur 54, 57, Benaras 55, Jabalpur 60, Agra 46)

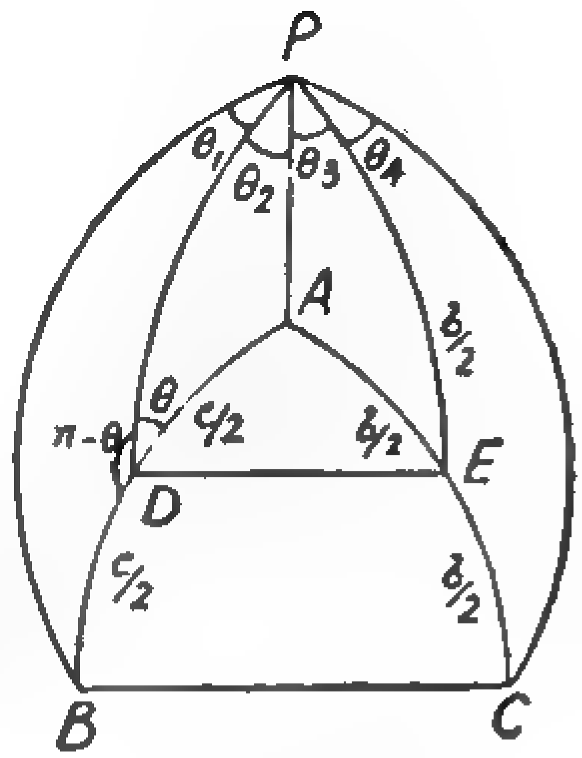


Fig. 48.

Since P is the pole of DE, therefore $DP=EP=\frac{\pi}{2}$. We have to prove

that $\qquad \angle BPC=2.\angle DPE,$

or $\quad \theta_1+\theta_2+\theta_3+\theta_4=2\,(\theta_2+\theta_3)$

or $\qquad \theta_1+\theta_4=\theta_2+\theta_3$

Applying the formula of consecutive four on ΔPDA choosing the four consecutive elements to be $AD=\frac{c}{2}$, θ, $PD=90°$ and θ_2,

$$\cos 90 \cos \theta=\sin 90 \cot \frac{c}{2}-\sin \theta \cot \theta_2$$

$$\therefore \quad \cot \frac{c}{2}=\sin \theta \cot \theta_2 \qquad \ldots(1)$$

Again from ΔPBD, applying the above formula, we have

$$\cot \frac{c}{2}=\sin (\pi-\theta) \cot \theta_1=\sin \theta \cot \theta_1. \qquad \ldots(2)$$

From (1) and (2), we get $\theta_1=\theta_2$

Similarly we can show that $\theta_4=\theta_3$,

from Δs PAE and PEC.

$\therefore \quad \theta_1+\theta_4=\theta_2+\theta_3$. Hence proved.

29. *If two angles of a spherical triangle be respectively equal to the sides opposite to them, show that the remaining side is supplement of the remaining angle or else that the triangle has two quadrants and two right angles and then the remaining side is equal to the remaining angle.*

We are given that $A=a$ and $B=b$, and we have to prove that

(i) $c=\pi-C$ or (ii) $a=b=(\pi/2)=A=B$ and $c=C$.

$$\cos c=\cos a \cos b+\sin a \sin b \cos C. \qquad \ldots(1)$$

$$\cos C=-\cos A \cos B+\sin A \sin B \cos c$$

$$= -\cos a \cos b + \sin a \sin b \cos C,$$

$$\because \quad A = a \text{ and } B = b. \qquad \ldots(2)$$

Adding (1) and (2), we get

$$\cos c + \cos C = \sin a \sin b (\cos C + \cos c)$$

or $$(\cos c + \cos C)(1 - \sin a \sin b) = 0$$

If $\cos c + \cos C = 0$, or $\cos c = -\cos C = \cos(\pi - C)$

$$\therefore \quad c = \pi - C,$$

i.e. the remaining side is supplement of the remaining angle.

If $1 - \sin a \sin b = 0$, then $\sin a \sin b = 1$.

Since $\sin \theta$ can never be greater than 1, hence we must have $a = b = \pi/2$.

From (1) by putting $a = b = \pi/2$ we get $\cos c = \cos C$,

$$\therefore \quad c = C.$$

and now $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = 1. \quad \because \quad c = C.$$

$$\therefore \quad \sin A = \sin a = 1, \quad \because \quad a = \pi/2, \quad \therefore \quad A = \pi/2.$$

$$\sin B = \sin b = 1, \quad \because \quad b = \pi/2, \quad \therefore \quad B = \pi/2.$$

Above shows that the triangle has two quadrants and two right angles and also the third side is equal to the third angle.

30 *In a spherical triangle whose sides are each less than $\pi/2$, prove that an exterior angle is greater than either of the interior opposite angles.* **(Agra 53, 56; Banaras 52)**

We are given that in spherical triangle ABC, AB, BC and CA are each less than $\pi/2$. Now produce AB and AC to meet at A'. Therefore

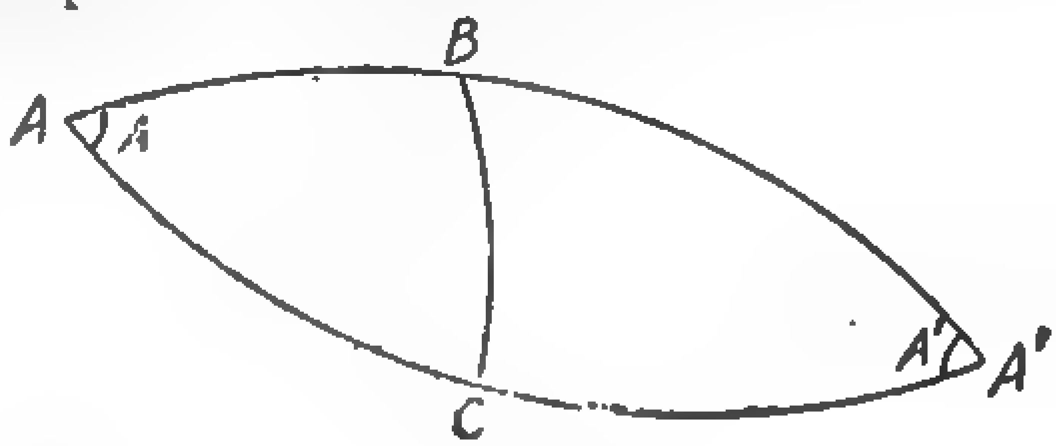


Fig 49

$$\angle A = \angle A'.$$

Clearly $A'B = \pi - AB$ *i.e.* $> \pi/2$ because $AB < \pi/2$

given.

Also $BC < \pi/2$ given. Hence in $\Delta A'CB$, $A'B > BC$. Hence the angles opposite to these sides will also satisfy the above inequality. Therefore $\angle BCA' > A'$ or $> A$

$$\because \quad A'=A.$$

Similarly we can prove that $\angle CBA'$ is also greater than $\angle A$. Hence proved.

31. *If in a triangle $a=b=\pi/3$ and $c=\pi/2$, prove that*

$$A+B+C=\pi+\cos^{-1}\tfrac{7}{9}. \qquad \textbf{(Agra 56)}$$

$$\sin A=\frac{2n}{\sin b \sin c}$$

$$=\frac{\{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c\}^{1/2}}{\sin b\sin c}$$

$$=\sqrt{\tfrac{2}{3}}=\sin B, \qquad \because \quad a=b=\pi/3 \text{ and } c=\pi/2.$$

Now $\dfrac{\sin A}{\sin a}=\dfrac{\sin C}{\sin c}$. $\therefore$ $\sin C=\dfrac{2\sqrt{2}}{3}$, $\because$ $\sin A=\sqrt{\tfrac{2}{3}}$.

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}=\frac{1}{\sqrt{3}}=\cos B,$$

and $\quad \cos C=-1/3.$

$$\sin(A+B)=\sin A\cos B+\cos A\sin B=\frac{2\sqrt{2}}{3},$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B=-1/3,$$

$$\cos(A+B+C)=\cos(A+B)\cos C-\sin(A+B)\sin C$$

$$=-\tfrac{1}{3}(-\tfrac{1}{3})-\frac{2\sqrt{2}}{3}\cdot\frac{2\sqrt{2}}{3}=-\tfrac{7}{9}$$

$$=\cos(\pi+\cos^{-1}\tfrac{7}{9}).$$

$$\therefore \quad A+B+C=\pi+\cos^{-1}\tfrac{7}{9},$$

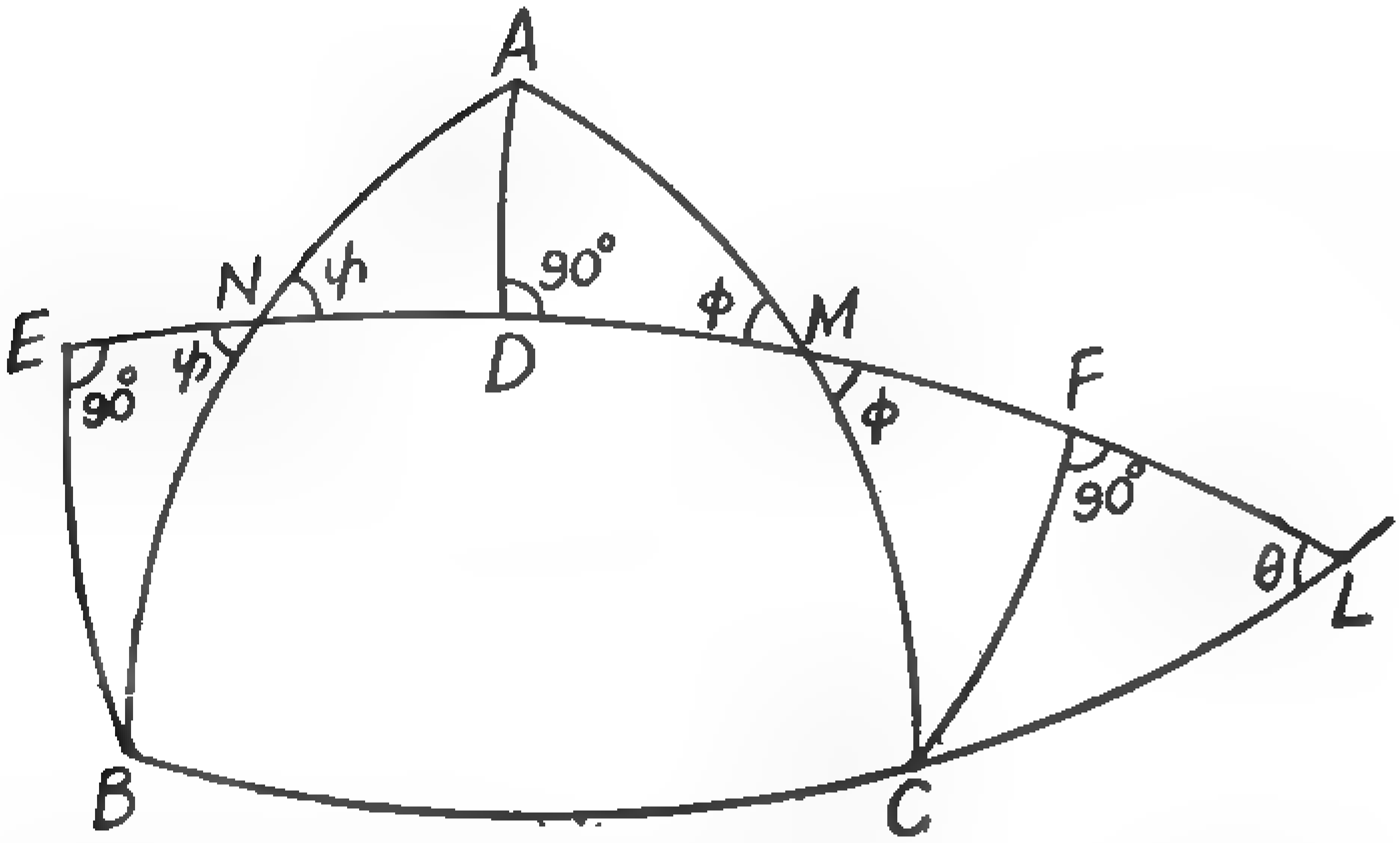
as sum of the angles is greater than two right angles, we have written $-\frac{7}{9}$ as $\cos(\pi+\cos^{-1}\frac{7}{9})$ and not $\cos(\pi-\cos^{-1}\frac{7}{9})$.

Note. See alternative method ahead in chapter V.

32. (a) *If a great circle intersects the sides BC, CA, AB of a spherial triangle in L, M and N respectively, prove that*

$$\frac{\sin BL}{\sin LC} \cdot \frac{\sin CM}{\sin MA} \cdot \frac{\sin AN}{\sin NB} = -1.$$

(Agra 51, Raj. 60)



LMN is a great circle which meets the sides BC, CA and AB in L, M and N respectively.

From A, B, C draw perpendiculars AD, BE and CF on the great circle LMN.

From right-angled triangle BEL, we get

$$\frac{\sin BE}{\sin BL} = \frac{\sin \theta}{\sin 90} = \sin \theta.$$

Again from right-angled triangle LFC, we get

$$\frac{\sin CF}{\sin CL} = \frac{\sin \theta}{\sin 90} = \sin \theta.$$

$$\therefore \quad \frac{\sin BE}{\sin BL} = \frac{\sin CF}{\sin CL},$$

$$\therefore \quad \frac{\sin BL}{\sin CL} = \frac{\sin BE}{\sin CF}. \qquad \ldots(1.$$

Again from right-angled triangle CFM,

$$\frac{\sin CF}{\sin CM} = \frac{\sin \phi}{\sin 90} = \sin \phi.$$

From right-angled triangle ADM,

$$\frac{\sin AD}{\sin AM}=\frac{\sin \phi}{\sin 90}=\sin \phi.$$

$$\therefore \quad \frac{\sin CF}{\sin CM}=\frac{\sin AD}{\sin AM}.$$

$$\therefore \quad \frac{\sin CM}{\sin AM}=\frac{\sin CF}{\sin AD} \qquad \ldots(2)$$

Again from right-angled triangles ADN and BEN,

$$\frac{\sin AD}{\sin AN}=\frac{\sin \psi}{\sin 90} \text{ and } \frac{\sin BE}{\sin BN}=\frac{\sin \psi}{\sin 90}.$$

$$\therefore \quad \frac{\sin AD}{\sin AN}=\frac{\sin BE}{\sin BN}$$

or

$$\frac{\sin AN}{\sin BN}=\frac{\sin AD}{\sin BE}. \qquad \ldots(2)$$

Multiplying (1), (2) and (3), we get

$$\frac{\sin BL}{\sin CL}\cdot\frac{\sin CM}{\sin AM}\cdot\frac{\sin AN}{\sin BN}=1.$$

Now $\sin CL=-\sin LC$, $\sin AM=-\sin MA$,

$\sin BN=-\sin NB$

i. e.

$$\frac{\sin BL}{\sin LC}\cdot\frac{\sin CM}{\sin MA}\cdot\frac{\sin AN}{\sin NB}=-1$$

Note. It is to be noted that while applying sine formula, everywhere we have taken the sides from the vertices A, B and C *i. e.* AD, AM; BE, BN; CL, CF. etc.

(b) *If three arcs be drawn from the angles of a spherical triangle through any point to meet the opposite sides, show that the products of the sines of the alternate segments of the sides are equal.*

Let P be any point. Through the angular points A, B, C arcs are drawn all of which pass through P and meet the opposite sides in D, E and F respectively.

$$\frac{\sin BP}{\sin BD}=\frac{\sin \alpha}{\sin \theta},$$

$$\frac{\sin CP}{\sin CD}=\frac{\sin (\pi-\alpha)}{\sin \phi}=\frac{\sin \alpha}{\sin \phi}.$$

$$\frac{\sin BP}{\sin BD}\cdot\frac{\sin CD}{\sin CP}=\frac{\sin \phi}{\sin \theta}. \quad \ldots(1)$$

Similarly,

$$\frac{\sin CP}{\sin CE}\cdot\frac{\sin AE}{\sin AP}=\frac{\sin \theta}{\sin \psi} \quad \ldots(2)$$

and $$\frac{\sin AP}{\sin AF}\cdot\frac{\sin BF}{\sin BP}=\frac{\sin \psi}{\sin \phi}. \quad \ldots(3)$$

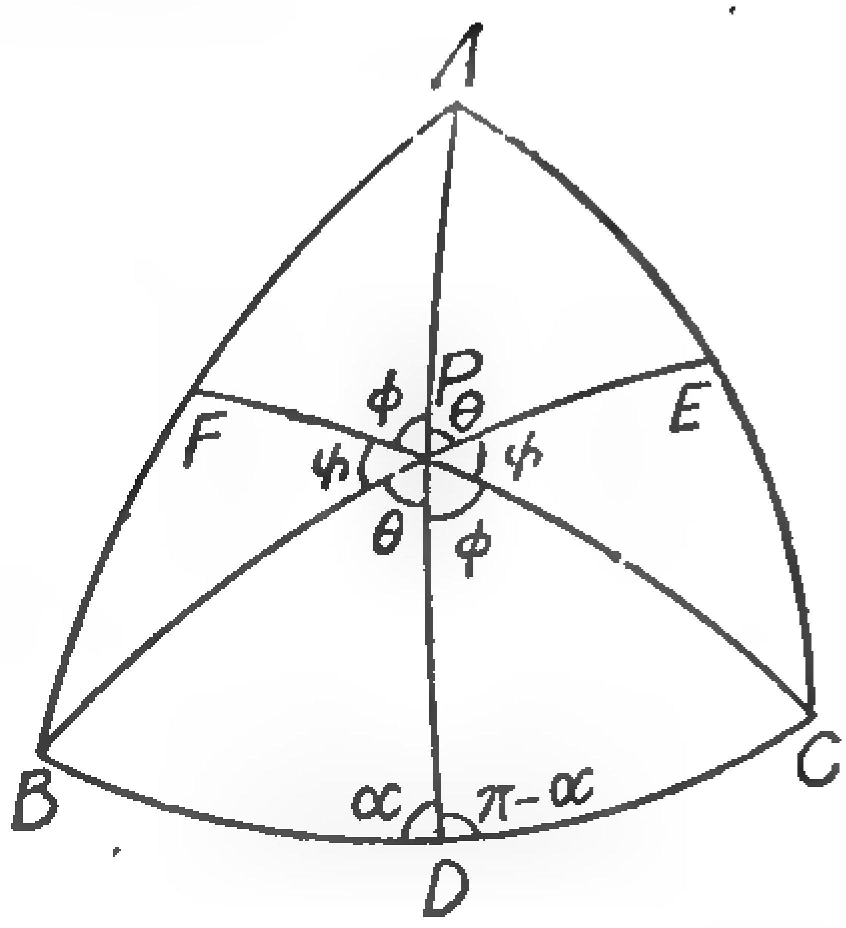


Fig.

Multiplying (1), (2) and (3), we get

$$\frac{\sin BP}{\sin BD}\cdot\frac{\sin CD}{\sin CP}\cdot\frac{\sin CP}{\sin CE}\cdot\frac{\sin AE}{\sin AP}\cdot\frac{\sin AP}{\sin AF}\cdot\frac{\sin BF}{\sin BP}=1$$

or $$\sin CD.\sin AE.\sin BF=\sin BD.\sin CE.\sin AF.$$

Proved.

33. *Prove that the Jacobian of the angles of a spherical triangle ABC with respect to the sides is numerically equal to* $\dfrac{\sin A}{\sin a}$.

(Rajputana 60)

$$\frac{\partial(A, B, C)}{\partial(a, b, c)}=\begin{vmatrix} \dfrac{\partial A}{\partial a} & \dfrac{\partial A}{\partial b} & \dfrac{\partial A}{\partial c} \\[2ex] \dfrac{\partial B}{\partial a} & \dfrac{\partial B}{\partial b} & \dfrac{\partial B}{\partial c} \\[2ex] \dfrac{\partial C}{\partial a} & \dfrac{\partial C}{\partial b} & \dfrac{\partial C}{\partial c} \end{vmatrix}$$

Now $\cos a = \cos b \cos c$
$+ \sin b \sin c \cos A.$ ...(1)

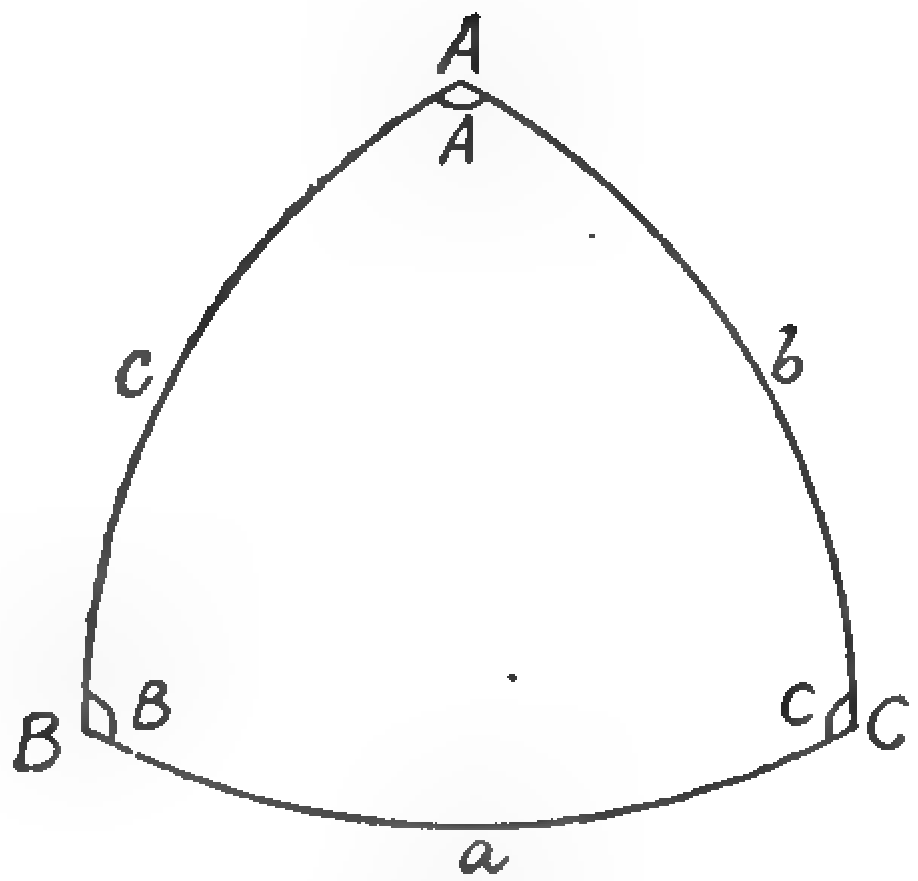


Fig. 49

Differentiating partially w. r. t. a we, get

$$-\sin a = -\sin b \sin c \sin A \frac{\partial A}{\partial a}.$$

$$\therefore \quad \frac{\partial A}{\partial a} = \frac{\sin a}{\sin b \sin c \sin A}$$

$$= \frac{k}{\sin b \ \sin c} \quad \text{...(2)}$$

where $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = k \text{ (say).} \quad \text{...(3)}$$

Again differentiating (1) partially w. r. t. b, we get

$$0 = (-\sin b \cos c + \cos b \sin c \cos A) - \sin b \sin c \sin A . \frac{\partial A}{\partial b}.$$

By sine cosine formula, we have

$$\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A.$$

$$\therefore \quad \sin a \cos C = -\sin b \sin c \sin A . \frac{\partial A}{\partial b}.$$

$$\therefore \quad \frac{\partial A}{\partial b} = -\frac{\sin a}{\sin b \sin c \sin A} . \cos C = -\frac{k \cos C}{\sin b \sin c}.$$

Similarly $$\frac{\partial A}{\partial c} = -\frac{k \cos B}{\sin b \sin c}$$

$$\therefore J = \frac{k^3}{\sin b \sin c . \sin c . \sin a . \sin a \sin b} \begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos C & -\cos A & 1 \end{vmatrix}$$

Expand the determinant

$$\therefore \quad J = \frac{k^3}{\sin^2 a \sin^2 b \sin^2 c} (1 - \cos^2 A - \cos^2 B - \cos^2 C$$
$$- 2 \cos A \cos B \cos C).$$

Now $\cos A = -\cos B \cos C + \sin B \sin C \cos a$

$$\therefore \quad \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

$$\therefore \quad \sin^2 a = 1 - \cos^2 a = \frac{\sin^2 B \sin^2 C - (\cos A + \cos B \cos C)^2}{\sin^2 B \sin^2 C}$$

or $\sin^2 a . \sin^2 B \sin^2 C = (1 - \cos^2 C)(1 - \cos^2 B) - (\cos^2 A$
$+ \cos^2 B \cos^2 C + 2 \cos A \cos B \cos C)$
$= 1 - \cos^2 A - \cos^2 B - \cos^2 C$
$- 2 \cos A \cos B \cos C$

$$\therefore \quad J = \frac{k^3}{\sin^2 a \sin^2 b \sin^2 c} . \sin^2 a \sin^2 B . \sin^2 C$$

$$= k^3 . \frac{\sin^2 B}{\sin^2 b} . \frac{\sin^2 C}{\sin^2 c} = k^3 . \frac{1}{k^2} . \frac{1}{k^2} = \frac{1}{k} = \frac{\sin A}{\sin a} \text{ [by (3)]}$$

$$\therefore \quad J = \frac{\sin A}{\sin a}.$$

Proved

CHAPTER III

RIGHT-ANGLED TRIANGLES

§ 1. If a spherical triangle be a right-angled triangle in which say angle $C=\pi/2$, then the corresponding formulae are obtained by putting $C=\pi/2$ in the formulae we have already proved for a general triangle as shown below. The general triangle has six elements whereas the right-angled triangle has five elements, the sixth being known to be $\pi/2$. All those formulae of the general triangle which do not involve the angle C will remain the same whereas those involving the element C will be changed as C will be put $\pi/2$ *i.e.* $\cos C=0$ and $\sin C=1$. The bold-typed formulae denote the corresponding formulae deduced from the general formulae written above them.

1. $\cos c=\cos a\cos b+\sin a\sin b\cos C$. (§ 1·9 P. 15)

$$\therefore \quad \mathbf{\cos c=\cos a\cos b}. \qquad \ldots(1)$$

2. $\cos A=-\cos B\cos C+\sin B\sin C\cos a$. (§ 2·1 P. 17)

$$\therefore \quad \mathbf{\cos A=\sin B\cos a}. \qquad \ldots(2)$$

3. $\cos B=-\cos C\cos A+\sin C\sin A\cos b$. (§ 2·2 P. 18)

$$\therefore \quad \mathbf{\cos B=\sin A\cos b}.$$

4. $\cos C=-\cos A\cos B+\sin A\sin B\cos c$. (§ 2·3 P. 18)

$$\therefore \quad \mathbf{\cos c=\cot A\cot B}.$$

5. $\dfrac{\sin A}{\sin a}=\dfrac{\sin B}{\sin b}=\dfrac{\sin C}{\sin c}$. (§ 3 P. 18)

$$\therefore \quad \mathbf{\sin a=\sin A\sin c}$$

and $\qquad \mathbf{\sin b=\sin B\sin c}.$

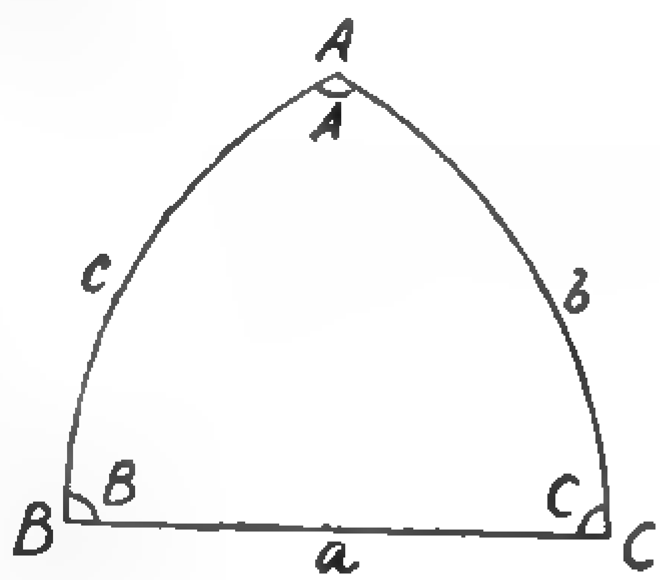


Fig. 50

6. Consecutive four *i.e* cotangent formula

$$\cos a\cos B=\sin a\cot c-\sin B\cot C \text{ (§ 8 P. 25)}$$

$\therefore$ **cos B = tan a cot c.**

7. $\cos b \cos A = \sin b \cot c - \sin A \cot C.$

$\therefore$ **cos A = tan b cot c.**

8. $\cos a \cos C = \sin a \cot b - \sin C \cot B.$

$\therefore$ **sin a = tan b cot B.**

9. $\cos b \cos C = \sin b \cot a - \sin C \cot A.$

$\therefore$ **sin b = tan a cot A.**

In the next article we shall give a very simple rule by the help of which all the formulae would follow very easily without the aid of the corresponding general formulae.

§ 2. Napier's Rule of circular parts $\angle C = \pi/2$.

The five elements of the right-angled triangle are marked as they are but outside the circle we have marked another set as follows. **The elements adjacent to the right angle, *i.e.* a and b, are written as they are, but for the remaining three elements**

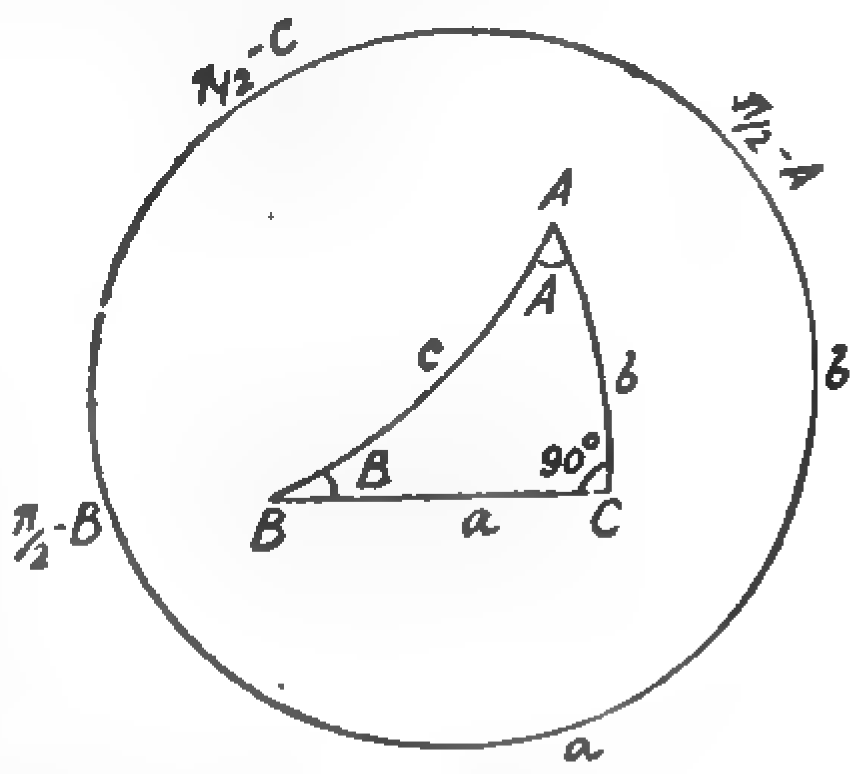


Fig. 51

***i.e.* B, c and A we have written their complements *i.e.* $\pi/2 - B$, $\pi/2 - c$, $\pi/2 - A$ outside the circle.**

Middle, adjacents and opposites.

If we choose any element and call it middle, then the two elements which are just next to it but on each side are called adjacent parts and the remaining two are called the opposite parts, *e.g.*, if we choose $\pi/2 - B$ as middle, then a and $\pi/2 - c$ will be called adjacents whereas b and $\pi/2 - A$ will be called opposite parts or if we choose 'a' as middle part then b and $\pi/2 - B$ will be called adjacents whereas $\pi/2 - c$ and $\pi/2 - A$ will be called opposite parts.

Napier's Rule is as under :

1. **sin of middle part**

   =product of tangents of adjacent parts.

2. **sin of middle part**

   =product of cosines of opposite parts.

The middle, adjacents and opposites are to be taken as explained. The above rule will give us all the formulae that we deduced from general formulae in § 1.

Taking $\pi/2-B$ as middle, we have

$\sin(\pi/2-B)=\tan a \tan(\pi/2-c)$ or $\cos B=\tan a \cot c$ (See § 1·6)

or $\sin(\pi/2-B)=\cos b \cos(\pi/2-A)$ or $\cos B=\sin A \cos b$ (See § 1·3)

Again taking $\pi/2-c$ as middle, we have

$$\sin(\pi/2-c)=\tan(\pi/2-A)\tan(\pi/2-B)$$

or $\cos c=\cot A \cot B$ (See § 1·4)

or $\sin(\pi/2-c)=\cos a \cos b$

or $\mathbf{\cos c=\cos a \cos b}$ **(V. Imp.)** [§ 1·1]

Similarly we can deduce any of the formulae of § 1.

Note. *While doing the question of right-angled triangle,, students should draw a triangle in which they should write the elements adjacent to right angle as they are and write complements of the remaining three and then apply Napier's rule as explained before.*

Exercise 2.

1. *In a spherical triangle ABC, in which $\angle C=\pi/2$, prove the following relations :*

(a) $\tan^2 \dfrac{a}{2}=\tan\dfrac{c+b}{2}\tan\dfrac{c-b}{2}$. **(Benaras 57, Delhi 56)**

$$\tan^2\frac{a}{2}=\frac{1-\cos a}{1+\cos a}.$$

Now in place of a we have to introduce c and b and as such we should search a relation between a, b and c and for this $\pi/2-c$ will have to be chosen as middle and a and b as opposites and applying Napier's rule, *i.e.* sine of middle = product of cosines of opposites, we get

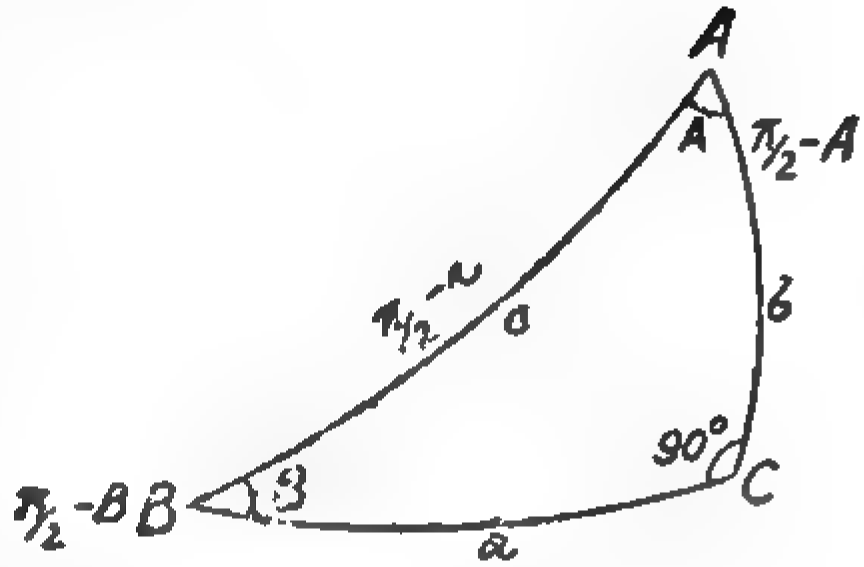


Fig. 52

$$\sin\left(\frac{\pi}{2}-c\right)=\cos a\cos b \quad \text{or} \quad \cos c=\cos a\cos b$$

or
$$\cos a=\frac{\cos c}{\cos b}.$$

Above relation is very important and will be used very frequently.

$$\therefore \quad \tan^2\frac{a}{2}=\frac{1-\dfrac{\cos c}{\cos b}}{1+\dfrac{\cos c}{\cos b}}=\frac{\cos b-\cos c}{\cos b+\cos c}$$

$$=\frac{2\sin\dfrac{+c}{2}\sin\dfrac{c-b}{2}}{2\cos\dfrac{b+c}{2}\cos\dfrac{b-c}{2}}=\tan\frac{b+c}{2}\tan\frac{c-b}{2}.$$

(b) $\tan^2\dfrac{A}{2}=\sin(c-b)\operatorname{cosec}(c+b)$. **(Benaras 55)**

$$\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}.$$

Now in place of A we want c and b. Therefore choosing $(\pi/2)-A$ as middle and b and $(\pi/2)-c$ as adjacents, we have by Napier's rule

$$\sin\left(\frac{\pi}{2}-A\right)=\tan b\tan\left(\frac{\pi}{2}-c\right) \text{ or } \cos A=\tan b\cot c.$$

$$\therefore \tan^2 \frac{A}{2} = \frac{1-\tan b \cot c}{1+\tan b \cot c} = \frac{\sin c \cos b - \cos c \sin b}{\sin c \cos b + \cos c \sin b}$$

$$= \frac{\sin (c-b)}{\sin (c+b)}.$$

(c) *$\sin a \tan (A/2) - \sin b \tan (B/2) = \sin (a-b)$* **(Sagar 59)**

$$\text{L.H.S.} = \sin a . \frac{1-\cos A}{\sin A} - \sin b . \frac{1-\cos B}{\sin B}.$$

Now $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{1}$ $\quad\because\quad C=\pi/2.$

$$\therefore \text{ L.H.S.} = \sin c \,[\cos B - \cos A]$$

$$= \sin c \,[\tan a \cot c - \tan b \cot c]$$

$$= \sin c . \frac{\cos c}{\sin c}\left[\frac{\sin a}{\cos a} - \frac{\sin b}{\cos b}\right]$$

$$= \cos c . \frac{\sin a \cos b - \cos a \sin b}{\cos a \cos b}$$

$$= \sin (a-b); \quad \because \quad \cos c = \cos a \cos b.$$

(d) *$\sin (c-a) = \sin b \cos a \tan (B/2) = \tan b \cos c \tan (B/2)$.*

(Rajputana 54, Lucknow 57)

You may proceed as above by putting the value of $\tan B/2$ etc.

or $\sin (c-a) = \sin c \cos a - \cos c \sin a$

$$= \frac{\sin b}{\sin B} . \cos a - \cos a \cos b . \tan b \cot B$$

$$= \cos a \sin b \left[\frac{1-\cos B}{\sin B}\right] = \cos a \sin b . \tan \frac{B}{2}.$$

Replacing $\cos a$ by $\dfrac{\cos c}{\cos b}$, we get the second formula.

(e) *$\dfrac{\sin (a-b)}{\sin (a+b)} = \tan \dfrac{A+B}{2} \tan \dfrac{A-B}{2}$.* **(Allahabad 1951)**

$$\tan \frac{A+B}{2} \tan \frac{A-B}{2} = \frac{\cos B - \cos A}{\cos B + \cos A} = \frac{\tan a \cot c - \tan b \cot c}{\tan a \cot c + \tan b \cot c}$$

$$= \frac{\sin a \cos b - \cos a \sin b}{\sin a \cos b + \cos a \sin b} = \frac{\sin (a-b)}{\sin (a+b)}.$$

(f) $\sin^2 \frac{c}{2} = \sin^2 \frac{a}{2} \cos^2 \frac{b}{2} + \cos^2 \frac{a}{2} \sin^2 \frac{b}{2}$.

(Nagpur 51, Delhi 60)

Multiply both sides by 4 and change in double angles.

$2(1-\cos c) = (1-\cos a)(1+\cos b) + (1+\cos a)(1-\cos b)$

$= 2(1-\cos a \cos b) = 2(1-\cos c)$:

$\therefore \quad \cos c = \cos a \cos b$.

(g) $\sin^2 a + \sin^2 b - \sin^2 c = \sin^2 a \sin^2 b$.

(Lucknow 55, Sagar 52)

L.H.S. $= \sin^2 a + \sin^2 b - 1 + \cos^2 c$ [Put $\cos c = \cos a \cos b$]

$= \sin^2 a + \cos^2 a \cos^2 b - (1-\sin^2 b)$

$= \sin^2 a + \cos^2 a \cos^2 b - \cos^2 b$

$= \sin^2 a - \cos^2 b\,(1-\cos^2 a)$

$= \sin^2 a - \cos^2 b \sin^2 a$

$= \sin^2 a\,(1-\cos^2 b) = \sin^2 a \sin^2 b$.

(h) $\cos^2 A + \cos^2 c - \cos^2 a = \cos^2 A \cos^2 c$.

L.H.S. $= \cos^2 A + \cos^2 c - 1 + \sin^2 a$

[Put $\sin a = \sin A . \sin c$]

$= \cos^2 A + \cos^2 c - 1 + \sin^2 A \sin^2 c$

$= \cos^2 A + \cos^2 c - 1 + (1-\cos^2 A)(1-\cos^2 c)$

$= \cos^2 A + \cos^2 c - 1 + 1 - \cos^2 A$

$- \cos^2 c + \cos^2 A \cos^2 c$

$= \cos^2 A \cos^2 c$.

(i) $\tan (A/2) \sin a = \sin c - \cos a \sin b$. **(Nagpur 54, 58)**

L.H.S. $= \tan \frac{A}{2} \sin a = \frac{1-\cos A}{\sin A} \sin a = \sin c\,[1-\cos A]$

by sine formula

$= \sin c - \sin c\,(\tan b \cot c)$

$= \sin c - \sin c . \frac{\sin b}{\cos b} \cdot \frac{\cos c}{\sin c}$

$= \sin c - \frac{\sin b}{\cos b} \cos a \cos b$

$= \sin c - \cos a \sin b$.

(j) $\sin(A+B)=\dfrac{\cos b+\cos a}{1+\cos b\cos a}$

(Allahabad 51, Sagar 55, Nagpur 60,

$\sin(A-B)=\dfrac{\cos b-\cos a}{1-\cos b\cos a}$. **(Delhi 58)**

Above follows from part (e) Q. 3 P. 41 by putting

$C=\pi/2$ and $\cos c=\cos a\cos b$.

(k) $\sin(c+a)\sin(c-a)=\sin^2 b\cos^2 a=\cos^2 A\sin^2 c$.

L.H S.$=\sin^2 c-\sin^2 a=\cos^2 a-\cos^2 c$

$=\cos^2 a-\cos^2 a\cos^2 b$

$=\cos^2 a(1-\cos^2 b)=\cos^2 a\sin^2 b$

or $=\sin^2 c-\sin^2 a$

$=\sin^2 c-\sin^2 c\sin^2 A$ [by sine formula]

$=\sin^2 c(1-\sin^2 A)=\sin^2 c\cos^2 A$.

(l) $\sin(A-a)\sin(A+a)=\sin^2 A\cos^2 c$.

L.H.S.$=\sin^2 A-\sin^2 a=\sin^2 A-\sin^2 A\sin^2 c$

$=\sin^2 A(1-\sin^2 c)=\sin^2 A\cos^2 c$.

(m) $\sin^2\left(45+\dfrac{c}{2}\right)=\dfrac{2\sin B}{\sin B-\sin b}$.

$$\cos^2\left(45+\frac{c}{2}\right)=\frac{1+\cos(90+c)}{2}=\frac{1-\sin c}{2}$$

$$=\frac{1-\dfrac{\sin b}{\sin B}}{2}=\frac{\sin B-\sin b}{2\sin B}\text{ etc.}$$

(n) $\sin s\sin(s-c)=\sin(s-a)\sin(s-b)$. **(Delhi 59)**

It follows by writing the value of $\tan C/2$, where $C=\pi/2$. (Page 21).

(o) $2\sin^2(c/2)=\sin^2\frac{1}{2}(a+b)+\sin^2\frac{1}{2}(a-b)$.

(Nagpur 57, Jabalpur 60)

$2(1-\cos c)=1-\cos(a+b)+1-\cos(a-b)$

or $2(1-\cos c)=2(1-\cos a\cos b)$

which is true as $\cos c=\cos a\cos b$.

(p) *Prove that in a spherical triangle ABC,*

$\tan b=\dfrac{\tan a\cos C+\tan c\cos A}{1-\tan a\tan c\cos A\cos C}$ **(Sagar 51)**

Draw $BD \perp$ to AC dividing it into two parts α and β such that $\alpha+\beta=b$.

$$\therefore \quad \tan b=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \quad ..(1)$$

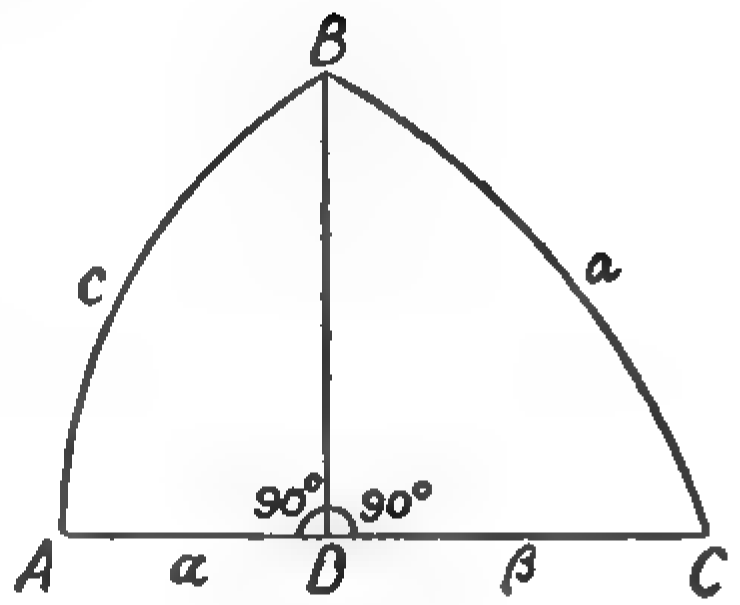


Fig. 53

Now $\cos A=\tan\alpha\cot c$.

and $\cos C=\tan\beta\cot a$

$\therefore \quad \tan\alpha=\cos A\tan c$

and $\tan\beta=\cos C\tan a$.

Putting these values of $\tan\alpha$ and $\tan\beta$ in (1), we get the required result.

Note. The question has already been done in Q. 9, P. 54 last exercise.

2. *If α, β be the arcs drawn from right angle C respectively perpendicular to and bisecting the hypotenuse c, show that*

(a) $\cos^2\alpha=\cos^2 A+\cos^2 B$.

CD is the arc drawn from C perpendicular to AB where $CD=\alpha$ and CE is the arc drawn from C bisecting AB, where $CE=\beta$.

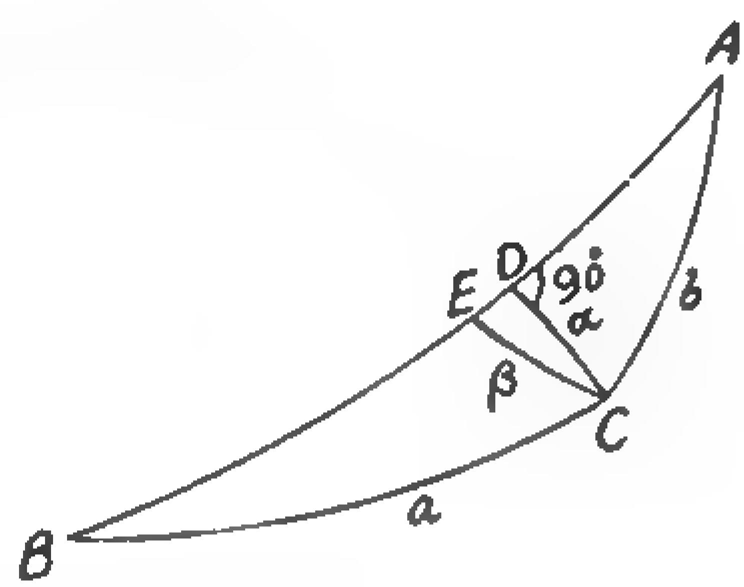


Fig. 54

Let $\angle ACD=\theta$, so that $\angle BCD=(\pi/2)-\theta$.

Now in right-angled triangle ACD choosing $(\pi/2)-A$ as middle, we have

$$\sin\{(\pi/2)-A\}=\cos\alpha.\cos\{(\pi/2)-\theta\}$$

$$\cos A=\cos\alpha\sin\theta. \quad ..(1)$$

Similarly from $\triangle BCD$ right-angled at D, in which $\angle BCD=(\pi/2)-\theta$, we have

$$\sin\{(\pi/2)-B\}=\cos\alpha.\cos[(\pi/2)-\{(\pi/2)-\theta\}]$$

or $$\cos B=\cos\alpha.\cos\theta. \quad ...(2)$$

Squaring and adding (1) and (2), we get

$$\cos^2 A+\cos^2 B=\cos^2\alpha.$$

(b) *$\cot^2 \alpha = \cot^2 a + \cot^2 b$.* **(Raj. 55, Utkal 56, Punjab 52)**

Choosing $\angle ACD$ as middle, we have from $\triangle ACD$,

$$\sin\{(\pi/2)-\theta\} = \tan\alpha \tan\{(\pi/2)-b\}$$

or $$\cos\theta = \tan\alpha \cot b \qquad \ldots(1)$$

Similarly from $\triangle BCD$, we have

$$\sin[(\pi/2)-\{(\pi/2)-\theta\}] = \tan\alpha \tan\{(\pi/2)-a\}$$

or $$\sin\theta = \tan\alpha \cot a \qquad \ldots(2)$$

Squaring (1) and (2), we get

$1 = \tan^2\alpha\,(\cot^2 a + \cot^2 b)$ or $\cot^2\alpha = \cot^2 a + \cot^2 b$.

(c) *$\sin^2\alpha \sin^2 c = \sin^2 a + \sin^2 b - \sin^2 c$.*

By sine formula we have $\dfrac{\sin\alpha}{\sin B} = \dfrac{\sin a}{\sin 90}$

or $$\sin\alpha = \sin a \sin B \qquad \ldots(1)$$

Again from $\triangle ABC$, $\dfrac{\sin B}{\sin b} = \dfrac{1}{\sin c}$ $\because$ $\angle C = (\pi/2)$.

$$\therefore \sin B = \frac{\sin b}{\sin c}, \quad \therefore \sin\alpha = \sin a \frac{\sin b}{\sin c} \text{ [by (1)]}$$

or $$\sin^2\alpha \sin^2 c = \sin^2 a \sin^2 b = \sin^2 a + \sin^2 b - \sin^2 c.$$

[by part (g) of P. 90)

(d) *$\sin^2\alpha = \tan AD \tan DB$.*

(Luck. 56, Nagpur 60, 61, Banaras 51)

Choosing CD as middle we have from $\triangle$s ACD and BCD,

$$\sin\alpha = \tan AD \tan\{(\pi/2)-\theta)\} = \tan AD \cot\theta.$$

$$\sin\alpha = \tan DB \tan[(\pi/2)-\{(\pi/2)-\theta\}] = \tan DB \tan\theta.$$

Multiplying, we get $\sin^2\alpha = \tan AD \tan DB$.

(e) *$\sin^2 a + \sin^2 b = 4\cos^2(c/2)\sin^2\beta$.* **(Lucknow 57, Nagpur 58, Delhi 58, Agra 48, 59, Sagar 52)**

If $\angle CEB = \phi$ then $\angle CEA = \pi - \phi$ and writing down the values of cos a and b and adding, we have as in Q. 1 P. 31,

$$\cos a + \cos b = 2\cos\beta\cos(c/2), \quad \because EB = BA = (c/2) \ldots(1)$$

or $$\sin^2\beta = 1 - \cos^2\beta = 1 - \frac{(\cos a + \cos b)^2}{4\cos^2(c/2)}.$$

or $4\cos^2(c/2)\sin^2\beta$

$$=2(1+\cos c)-(\cos^2 a+\cos^2 b+2\cos a\cos b)$$
$$=2-\cos^2 a-\cos^2 b,\ \because\ \cos c=\cos a\cos b,$$
$$=(1-\cos^2 a)+(1-\cos^2 b)=\sin^2 a+\sin^2 b. \qquad \ldots(2)$$

(f) $\cot^2\beta=\dfrac{(\cos a+\cos b)^2}{\sin^2 a+\sin^2 b}.$

It follows by dividing square of (1) by (2).

(g) $\sin^2(c/2)\,(1+\sin^2\alpha)=\sin^2\beta.$

(Agra 53, 58, 60 ; Delhi 60)

We have proved from part (c),

$$\sin\alpha=\frac{\sin a\sin b}{\sin c}.$$

$\therefore\ \sin^2(c/2)\,(1+\sin^2\alpha)=\sin^2(c/2)$

$$\times\left[\frac{\sin^2 c+\sin^2 a\sin^2 b}{\sin^2 c}\right]$$

$$=\sin^2(c/2).\left[\frac{(1-\cos^2 c)+(1-\cos^2 a)(1-\cos^2 b)}{4\sin^2(c/2)\cos^2(c/2)}\right]$$

$$=\frac{1}{4\cos^2(c/2)}[1-\cos^2 a\cos^2 b+1-\cos^2 a-\cos^2 b$$
$$+\cos^2 a\cos^2 b]$$

$$=\frac{\sin^2 a+\sin^2 b}{4\cos^2(c/2)}=\sin^2\beta \qquad \text{[by (2) of part (e)].}$$

Independent Proof of part (g).

Since $\angle C=(\pi/2)$,

$\therefore\ \cos c=\cos a\cos b\ \ldots(1)$

$\angle CDB=(\pi/2)$,

$\therefore\ \cos a=\cos\alpha\cos\{(c/2)-x\}.$

$\angle CDA=(\pi/2)$,

$\therefore\ \cos b=\cos\alpha\cos\{(c/2)+x\}.$

$\therefore\ \cos a\cos b$

$$=\cos^2\alpha\,\{\cos^2(c/2)-\sin^2 x\}$$
$$=\cos^2\alpha\,\{\cos^2 x-\sin^2(c/2)\}$$

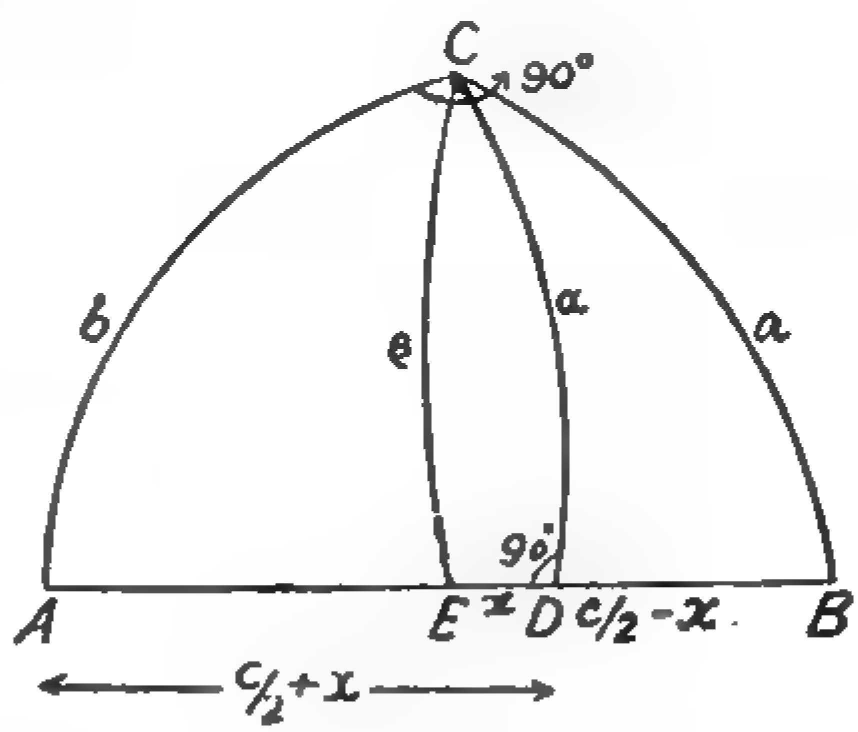


Fig. 55

$\because$ $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

or $\cos c=\{\cos^2\alpha\cos^2 x-\cos^2\alpha\sin^2(c/2)\}$ [by (1)].

Now from right-angled $\triangle\, CED$, $\cos\beta=\cos\alpha\cos x$.

$\therefore$ $1-2\sin^2(c/2)=\cos^2\beta-\cos^2\alpha\ \sin^2(c/2)$

or $$1-\cos^2\beta=\sin^2(c/2)\,(2-\cos^2\alpha)$$
$$=\sin^2(c/2)\,(1+1-\cos^2\alpha)$$

or $$\sin^2\beta=\sin^2(c/2)\,(1+\sin^2\alpha).$$ Hence proved.

2. *If the side C be a quadrant and δ the length of the arc perpendicular to it from C, show that*

(*a*) $\cos^2\delta=\cos^2 a+\cos^2 b.$ **(Utkal 59)**

If $AD=\alpha$ then $BD=\pi/2-\alpha$ as $AB=\pi/2$.

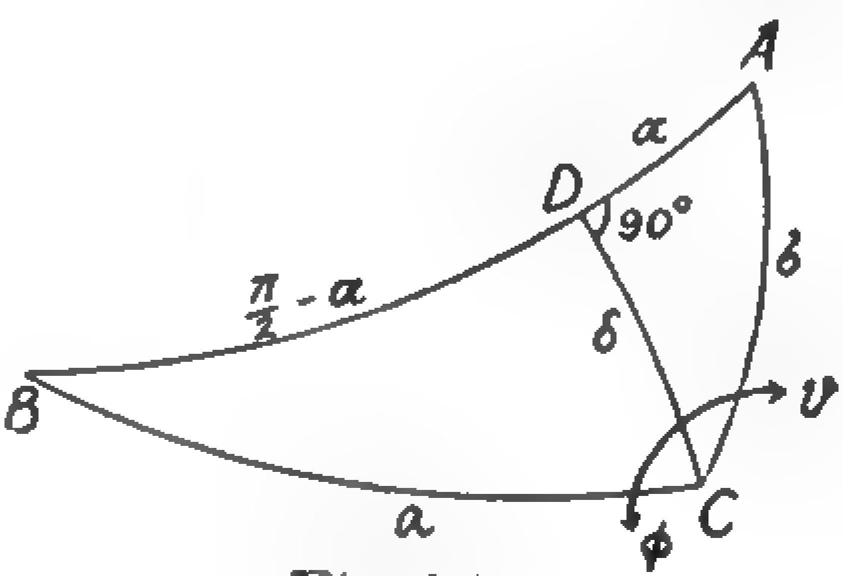


Fig. 56

From right-angled triangle ABC, we have

$$\cos b=\cos\alpha\cos\delta$$

and similarly from $\triangle BCD$,

$$\cos a=\cos\{(\pi/2)-\alpha\}\cos\delta$$
$$=\sin\alpha\cos\delta.$$

Squaring and adding,

$$\cos^2 a+\cos^2 b=\cos^2\delta\,(\cos^2\alpha+\sin^2\alpha)=\cos^2\delta.$$ Proved.

(b) $\cot^2\delta=\cot^2 A+\cot^2 B.$

$$\sin\alpha=\cot A\tan\delta \quad [\text{from } \triangle\, ADC].$$
$$\sin\{(\pi/2)-\alpha\}=\cot B\tan\delta \quad [\text{from } \triangle\, BDC].$$

Squaring and adding, we get

$$\sin^2\alpha+\cos^2\alpha=(\cot^2 A+\cot^2 B)\tan^2\delta$$

or $$\cot^2\delta=\cot^2 A+\cot^2 B.$$

(c) $\sin^2\delta=\cot\theta\cot\phi$ *where θ, φ are the segments of the angle C.* **(Utkal 57)**

$$\sin\delta=\cot\theta.\tan\alpha \quad [\text{from } \triangle\, ADC],$$
$$\sin\delta=\cot\phi.\cot\alpha \quad [\text{from } \triangle\, BDC].$$

Multiplying, we get

$$\sin^2\delta=\cot\theta\cot\phi.$$

4. (a) *If $C=90°$ and $2S=A+B+C$, prove that*

$$\frac{\sin^2 a}{\cos a}+\frac{\sin^2 b}{\cos b}+\frac{\sin^2 c}{\cos c}=2\frac{\sin^2 c}{\cos c}\sin^2 S.$$

or
$$\frac{\sin^2 a\cos b+\sin^2 b\cos a}{\cos a\cos b}=-\frac{\sin^2 c}{\cos c}(1-2\sin^2 S)$$

$$=-\frac{\sin^2 c}{\cos c}\cos 2S.$$

$$2S=A+B+\pi/2, \quad \because \quad \angle C=\pi/2.$$

$\therefore \quad \cos 2S=-\sin(A+B)$ and $\cos c=\cos a\cos b$.

$\therefore \quad (1-\cos^2 a)\cos b+(1-\cos^2 b)\cos a=\sin^2 c\sin(A+B)$.

$$\frac{(\cos a+\cos b)-\cos a\cos b(\cos a+\cos b)}{\sin^2 c}$$
$$=\sin A\cos B+\cos A\sin B.$$

Now using sin formula with $\angle C=\pi/2$ and $\cos c=\cos a\cos b$,

$$\frac{(\cos a+\cos b)(1-\cos c)}{\sin^2 c}=\frac{\sin a}{\sin c}\cdot\frac{\cos b-\cos a\cos c}{\sin a\sin c}$$
$$+\frac{\sin b}{\sin c}\cdot\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

or
$$\frac{(\cos a+\cos b)(1-\cos c)}{\sin^2 c}=\frac{(\cos a+\cos b)-\cos c(\cos a+\cos b)}{\sin^2 c}$$
$$=\frac{(\cos a+\cos b)(1-\cos c)}{\sin^2 c}$$

$\therefore$ L H S = R H S. Hence proved.

(b) *In a spherical triangle ABC, the angle $C=120°$; show that if the arc of a great circle drawn through C to meet AB at right angles is $\tan^{-1}\sqrt{3}/2$, then*

$$\cot^2 a+\cot^2 b+\cot a\cot b=1.$$

We are given that arc

$$CN=x=\tan^{-1}\sqrt{3}/2.$$

$$\therefore \quad \tan x=\sqrt{3}/2.$$

Let $\angle ACN=\theta$

so that $\angle BCN=120-\theta$.

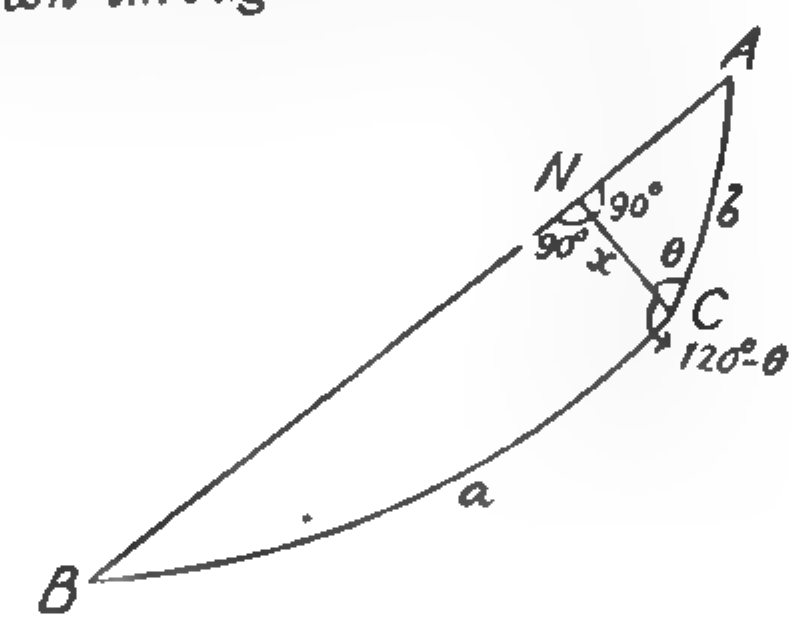


Fig. 43

Applying sine of the middle = product of tangents of adjacents on Δs ACN and BCN.

Choosing θ and $120°-\theta$ as middle parts

$$\cos\theta=\tan x \cot b \qquad \ldots(2)$$

$$\cos(120°-\theta)=\tan x \cot a. \qquad \ldots(3)$$

We have to eliminate θ between (2) and (3).

$\cos 120 \cos\theta+\sin 120 \sin\theta=\tan x \cot a$

or $\quad -\frac{1}{2}\cos\theta+\sqrt{3}/2 \sin\theta=\tan x \cot a$

or $\quad \frac{1}{2}\sqrt{3}\sin\theta=\frac{1}{2}\tan x \cot b+\tan x \cot a \qquad$ [by (2)]

Put $\tan x=\sqrt{3}/2$ and square both sides.

or $\quad \frac{3}{4}(1-\cos^2\theta)=\frac{3}{4}(\frac{1}{4}\cot^2 b+\cot a \cot b+\cot^2 a)$

or $\quad 1-\tan^2 x \cot^2 b=\frac{1}{4}\cot^2 b+\cot a \cot b+\cot^2 a$

or $\quad 1=\cot^2 b+\cot a \cot b+\cot^2 a \qquad$ [by (1).]

5. *If O be the point of intersection of arcs AD, BE, CF drawn from the angles of a triangle perpendicular to the opposite sides meeting them at D, E, F respectively, show that*

$$\frac{\tan AD}{\tan OD}, \quad \frac{\tan BE}{\tan OE}, \quad \frac{\tan CF}{\tan OF}$$

are respectively equal to

$$1+\frac{\cos A}{\cos B\cos C}, \quad 1+\frac{\cos B}{\cos C\cos A}, \quad 1+\frac{\cos C}{\cos A\cos B}.$$

$\sin BD=\tan AD \cot B$

[from Δ ABD],

$\sin BD=\tan OD \cot\theta$

[from Δ OBD].

$$\therefore \quad \frac{\tan AD}{\tan OD}=\frac{\tan B}{\tan\theta}$$

$$=\tan B\cot\theta \ldots(1)$$

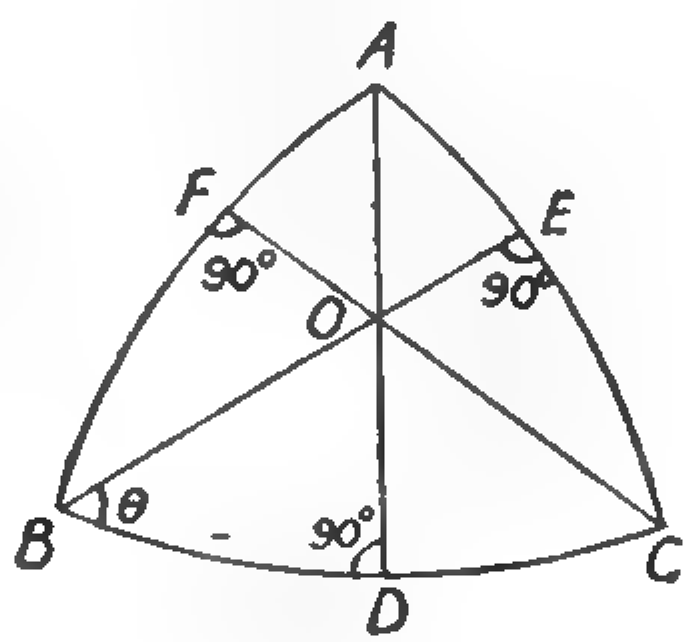


Fig. 57

Now $\sin\{(\pi/2)-BC\}$

$=\tan\{\pi/2)-\theta\}\tan\{(\pi/2)-C\}$

[from Δ BEC]

or $\quad \cos a=\cot\theta\cot C; \quad \therefore \quad \cos a \tan C=\cot\theta.$

Putting in (1), we get

$$\frac{\tan AD}{\tan OD}=\tan B \tan C \cos a$$
$$=\frac{\sin B \sin C}{\cos B \cos C}\left(\frac{\cos A+\cos B \cos C}{\sin B \sin C}\right)$$
$$=\frac{\cos A}{\cos B \cos C}+1.$$

Similarly we can prove other part.

(b) *Prove that*

$\tan BD \tan CE \tan AF=\tan LC \tan EA \tan FB$.

(Bihar 60, Nagpur 54, 58, Agra 1952, 57, 59, 61)

$\sin OD=\tan BD \cot BOD$ [from $\triangle BOD$]
$\sin OD=\tan DC \cot DOC$ [from $\triangle COD$].

$$\therefore \quad \frac{\tan BD}{\tan DC}=\frac{\tan BOD}{\tan DOC}.$$

Similarly $\dfrac{\tan CE}{\tan EA}=\dfrac{\tan COE}{\tan EOA}$ and $\dfrac{\tan AF}{\tan FB}=\dfrac{\tan AOF}{\tan FOB}$.

Multiplying and noting that $\angle BOD=\angle EOA$ etc., we get the required result.

6. *If two great circular arcs are drawn from C, one perpendicular to AB and other bisecting angle C and ϕ be the angle between them, show that* $\tan \phi=\dfrac{\cos \dfrac{a-b}{2}}{\cos \dfrac{a+b}{2}} \tan \dfrac{A-B}{2}$.

In $\triangle BCD$ right angled at D, $\angle BCD=\dfrac{C}{2}+\phi$;

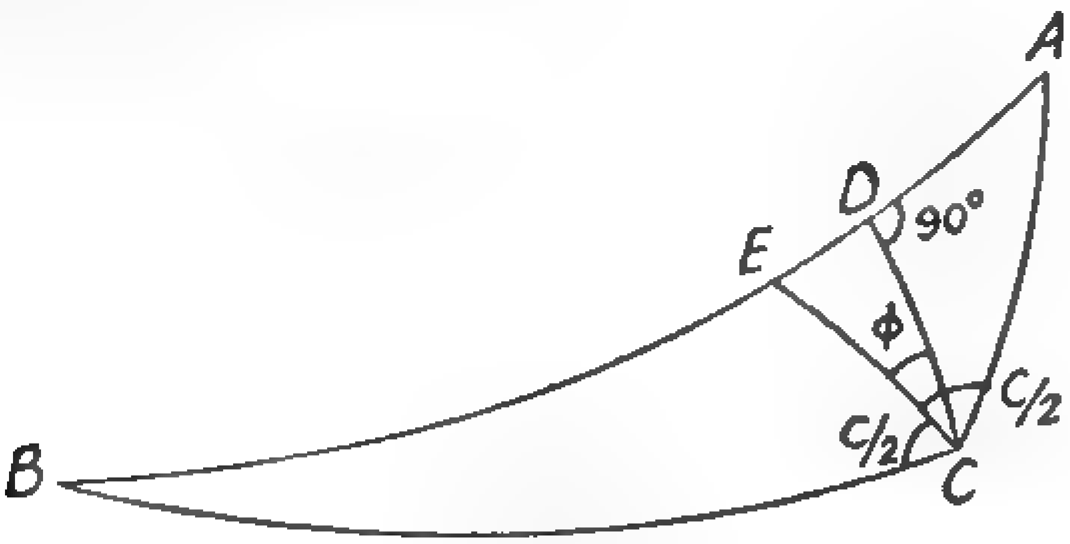


Fig. 58.

$$\therefore \quad \sin\left(\frac{\pi}{2}-B\right)=\cos CD \cos\left\{\frac{\pi}{2}-\left(\frac{C}{2}+\phi\right)\right\}$$

or
$$\cos B=\cos CD \sin\left(\frac{C}{2}+\phi\right);$$

Similarly from right-angled triangle ACD in which $\angle ACD=\frac{C}{2}-\phi$, we have

$$\cos A=\cos CD \sin\left(\frac{C}{2}-\phi\right);$$

$\therefore \quad \dfrac{\cos B}{\cos A}=\dfrac{\sin\left(\frac{C}{2}+\phi\right)}{\sin\left(\frac{C}{2}-\phi\right)}$. Apply componendo and dividendo.

$$\frac{\cos B-\cos A}{\cos B+\cos A}=\frac{\sin\left(\frac{C}{2}+\phi\right)-\sin\left(\frac{C}{2}-\phi\right)}{\sin\left(\frac{C}{2}+\phi\right)+\sin\left(\frac{C}{2}-\phi\right)}$$

or
$$\frac{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}=\frac{2\cos\frac{C}{2}\sin\phi}{2\sin\frac{C}{2}\cos\phi}$$

or
$$\tan\frac{A+B}{2}\tan\frac{A-B}{2}=\tan\phi.\cot\frac{C}{2}. \qquad \ldots(1)$$

Now by Napier's Analogy, $\tan\dfrac{A+B}{2}=\dfrac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\dfrac{C}{2}$.

Putting the value of $\tan\dfrac{A+B}{2}$ in (1) and cancelling

$\cot \dfrac{C}{2}$ we get the result.

7. *ABC is a great circle of a sphere; AA′, BB′, CC′ are arcs of great circles drawn at right angles to ABC and reckoned positive when they lie on the same side of it; show that the condition that A′, B′, C′ should lie in a great circle is*

$$\tan AA' \sin BC + \tan BB' \sin CA + \tan CC' \sin AB = 0.$$

(Nagpur 56, Agra 48, Pb. 55, Sagar 59, Raj 59)

In order that A', B', C' lie on a great circle the arc $C'B'A'$ when produced must meet the great circle ABC. Let the two great circles meet at O such that $\angle A'OA = \theta$.

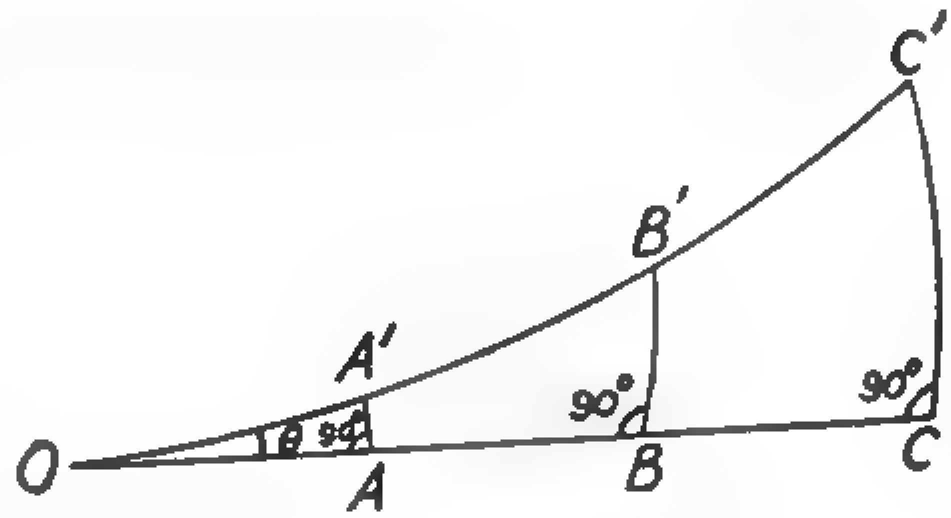


Fig. 59

$$\sin OA = \tan AA' \cot \theta,$$
$$\sin OB = \tan BB' \cot \theta,$$
$$\sin OC = \tan CC' \cot \theta;$$

$\therefore \quad \tan AA' \sin BC + \tan BB' \sin CA + \tan CC' \sin AB$

$= \tan \theta \{\sin OA \sin BC + \sin OB \sin CA + \sin OC \sin AB\}$

$= \tan \theta \{\sin (OB - AB) \sin BC + \sin OB \sin CA$
$\qquad + \sin (OB + EC \ \sin AB\}$

$= \tan \theta \{\sin OB \cos AB \sin BC - \cos OB \sin AB \sin BC$
$\qquad + \sin OB \sin GA + \sin OB \cos BC \sin AB$
$\qquad + \cos OB \sin BC \sin AB\}$

[cancel 2nd and last term and take $\sin OB$ common]

$= \tan \theta \sin OB \{\sin BC \cos AB + \cos BC \sin AB + \sin CA\}$

$= \tan \theta \sin OB \{\sin (BC + AB) + \sin CA\}$

$= \tan \theta \sin OB \{\sin AC + \sin CA\}$

$= \tan \theta \sin OB \{\sin AC - \sin AC\} = 0.$ Hence proved.

8. (a) *In a spherical triangle if* $A=\frac{\pi}{5}$, $B=\frac{\pi}{3}$, *and* $C=\frac{\pi}{2}$, *show that* $a+b+c=\frac{\pi}{2}$.

[Nagpur 59, Agra 52, 55, Sagar 57, Utkal 55, 59, Agra 61, Punjab 52]

By Napiers rule we have from right-angled triangle ABC,

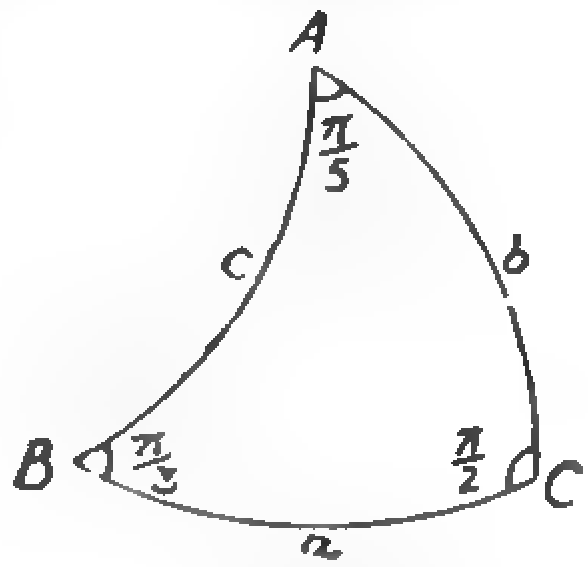


Fig. 60

$$\cos\frac{\pi}{3}=\tan a\cot c$$

and $$\cos\frac{\pi}{5}=\tan b\cot c.$$

$\therefore$ $(\tan a+\tan b)\cot c$

$$=\tfrac{1}{2}+\frac{\sqrt{5}+1}{4}=\frac{3+\sqrt{5}}{4}$$

or $$\frac{\sin a\cos b+\cos a\sin b}{\cos a\cos b}.\cot c=\frac{3+\sqrt{5}}{4}$$

or $$\frac{\sin(a+b)}{\cos a\cos b}.\cot c=\frac{3+\sqrt{5}}{4}. \qquad \ldots(1)$$

Now we shall find the value of $\cot c$.

$$\cos c=\cot\frac{\pi}{5}\cot\frac{\pi}{3}=\frac{1}{\sqrt{3}}\cot\frac{\pi}{5};$$

$$\therefore \quad \sec c=\sqrt{3}\tan\frac{\pi}{5}.$$

$$\therefore \quad \tan^2 c=\sec^2 c-1=3\tan^2\frac{\pi}{5}-1=3\left(\sec^2\frac{\pi}{5}-1\right)-1$$

or $$\tan^2 c=3\sec^2 36^\circ-4=3.\frac{16}{(\sqrt{5}+1)^2}-4$$

$$=\frac{48-4(6+2\sqrt{5})}{(\sqrt{5}+1)^2}=\frac{4(6-2\sqrt{5})}{(\sqrt{5}+1)^2}=\left(2.\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^2$$

$$\therefore \quad \tan c=2.\frac{\sqrt{5}-1}{\sqrt{5}+1},$$

$$\therefore \quad \cot c = \frac{1}{2} \cdot \frac{\sqrt{5}+1}{\sqrt{5}-1} = \frac{1}{2} \frac{(\sqrt{5}+1)^2}{5-1} = \frac{3+\sqrt{5}}{4},$$

Putting the value of $\cot c$ in (1) and replacing $\cos a \cos b$ by $\cos c$, $\because \quad \angle C = \pi/2$,

$$\therefore \quad \frac{\sin(a+b)}{\cos c} \cdot \frac{3+\sqrt{5}}{4} = \frac{3+\sqrt{5}}{4}$$

or $$\sin(a+b) = \cos c = \sin(\pi/2 - c)$$

or $$a+b = \pi/2 - c \quad \text{or} \quad a+b+c = \pi/2.$$

Hence proved.

(b) *If ABC be a spherical triangle right-angled at C and $\cos A = \cos^2 a$, show that if A be not a right angle, $b+c=\pi/2$ or $3\pi/2$ accoeding as b and c are both less or both greater than $\pi/2$.*

[Nagpur 58, Agra 57]

Since $C = \pi/2$, $\therefore \cos c = \cos a \cos b$.

and $$\cos A = \cos^2 a \text{ (given)}. \qquad \ldots(1)$$

Also $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

or $$\cos^2 a = \frac{\cos a - \cos b . \cos a \cos b}{\sin b \sin c}; \quad \text{[by (1)]}$$

$$\therefore \quad \cos a = \frac{1-\cos^2 b}{\sin b \sin c} = \frac{\sin b}{\sin c} \quad \text{or} \quad \frac{\cos c}{\cos b} = \frac{\sin b}{\sin c};$$

$$\therefore \quad \sin c \cos c = \sin b \cos b \quad \text{or} \quad \sin 2b = \sin 2c;$$

$$\therefore \quad 2b = 2c \quad \text{or} \quad \pi - 2c \quad \text{or} \quad 3\pi - 2c.$$

If $2b = 2c$, then $b = c$, $\therefore$ $B = C$, but $C = \pi/2$, therefore $B = \pi/2$, and a triangle having two right-angles is not possible.

If $2b = \pi - 2c$ then $b + c = \pi/2$. This will hold good when both b and c are less than $\pi/2$. We cannot say that $b = \pi/2$ and $c = 0$ as $c = 0$ is meaningless.

If $2b = 3\pi - 2c$, then $b + c = 3\pi/2$. As any cf the sides of a spherical triangle is not greater than π, hence the above will hold good if both b and c are greater than $\pi/2$.

(c) *If $2s=a+b+c$ and $\angle C=\pi/2$, prove that*

$$\sin s \sin(s-c)=\sin(s-a)\sin(s-b).$$

$$\tan\frac{C}{2}=\left\{\frac{\sin(s-a)\sin(s-b)}{\sin s.\sin(s-c)}\right\}^{1/2}=\tan\frac{\pi}{4}=1 \text{ etc.}$$

9. (a) *OX and OY are two great circles of a sphere at right angles to each other. P is any point in AB another great circle. OC $(=p)$ is the arc perpendicular to AB from O, making the angle COX $(=\alpha)$ with OX PM, PN are arcs perpendicular to OX, OY respectively. Show that if $OM=x$ and $ON=y$,*

$$\cos\alpha\tan x+\sin\alpha\tan y=\tan p.$$ **[Punjab 43, Sagar 56, Agra 53, Rajputana 54, Vikram 59, Nagpur 60]**

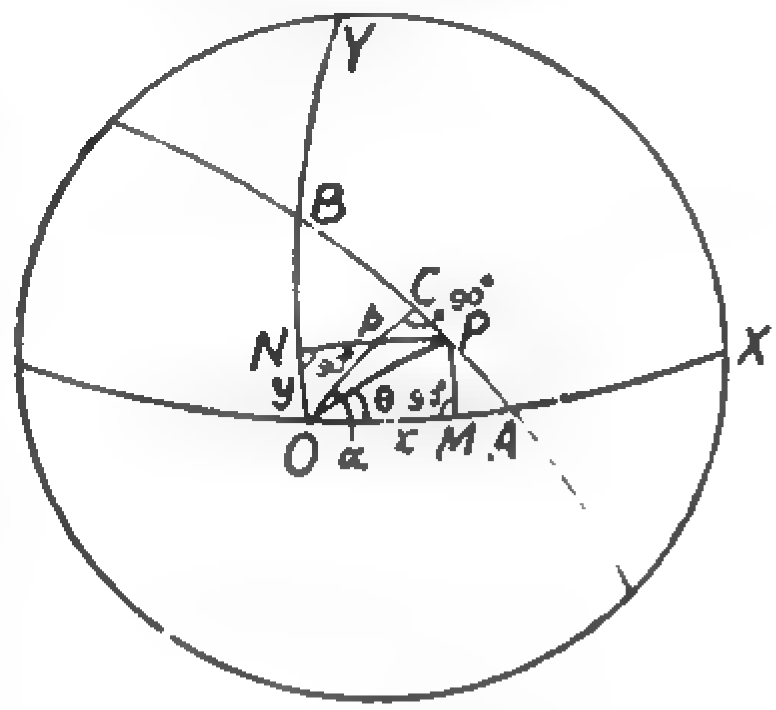


Fig. 61

Let $OP=r$ and $\angle POA=\theta$.

From right angled triangles OMP, ONP and OPC, we have

$$\cos\theta=\tan x\cot r, \qquad \ldots(1)$$

$$\sin\theta=\tan y\cot r \qquad \ldots(2)$$

$$\because\quad \angle PON=\pi/2-\theta.$$

As the two great circles OX and OY are at right angles

and $$\cos(\alpha-\theta)=\tan p\cot r$$

or $$\cos\alpha\cos\theta+\sin\alpha\sin\theta=\tan p\cot r. \qquad \ldots(3)$$

Putting the values of $\cos\theta$ and $\sin\theta$ in (1), we get

$$\cot r\{\cos\alpha\tan x+\sin\alpha\tan y\}=\tan p\cot r$$

or $$\cos\alpha\tan x+\sin\alpha\tan y=\tan p.$$ Hence proved.

(b) *The position of a point on a sphere with reference to two great circles at right angles to each other as axes is determined by the portions θ, ϕ of these circles cut off by great circles through the points and through two points on the axes each $\pi/2$ from their point of intersection. Show that if their points (θ, ϕ), (θ', ϕ'), (θ'', ϕ'')*

lie on the same great circle,

$$\tan\phi\,(\tan\theta'-\tan\theta'')+\tan\phi'\,(\tan\theta''-\tan\theta)$$
$$+\tan\phi''\,(\tan\theta-\tan\theta')=0.$$

[Delhi 59, Lucknow 40]

Let X and Y be two points on the two great circles intersecting at right angles at O such that $OX=OY=\pi/2$.

Let P be any point on the sphere. Join XP and YP to meet the great circles OY and OX in N and M respectively, so that $OM=\theta$ and $ON=\phi$.

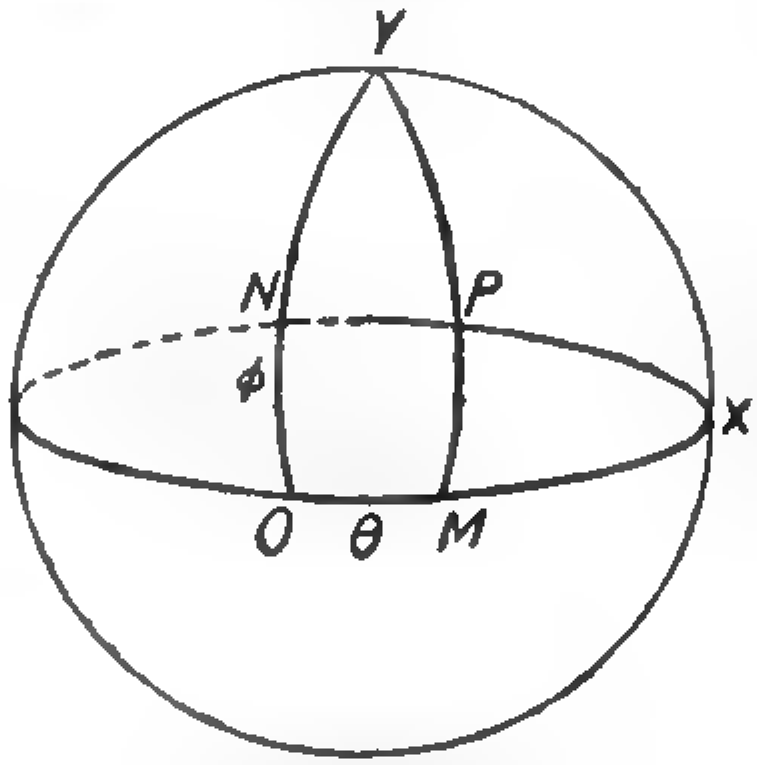


Fig. 62

Now clearly X is the pole of great circle OX, therefore PN is $\perp$ to OY and similarly PM is perpendicular to OX. Again if p be the length of perpendicular from O on the great circle on which P lies and α the angle it makes with OX, then as proved in part (a),

$$\cos\alpha.\tan\theta+\sin\alpha\tan\phi=\tan p. \qquad \ldots(1)$$

Here $\theta=x$ and $\phi=y$ of part (a).

Similarly $\cos\alpha.\tan\theta'+\sin\alpha\tan\phi'=\tan p \qquad \ldots(2)$

and $\cos\alpha\tan\theta''+\sin\alpha\tan\phi''=\tan p, \qquad \ldots(3)$

as (θ', ϕ') and (θ'', ϕ'') lie on the same great circle on which P *i.e.* (θ, ϕ) lies.

Eliminating α and p, we have

$$\begin{vmatrix} \tan\theta & \tan\phi & 1 \\ \tan\theta' & \tan\phi' & 1 \\ \tan\theta'' & \tan\phi'' & 1 \end{vmatrix}=0.$$ The determinant when expanded gives the result.

(c) *If a point on a sphere referred to two great circles at right angles to each other as axes, by means of the portions of these axes*

cut off by great circles drawn through the point and the points on the axes each 90° from their intersection, show that the equation to a great circle is $\tan\theta \cot\alpha + \tan\phi \cot\beta = 1$.

(Nagpur 47, Benares 57)

We have proved in part (a) that

$\cos\psi \tan\theta + \sin\psi \tan\phi = \tan p$

where p is the arc perpendicular to AB from O and ψ the angle which the perpendicular makes with OX.

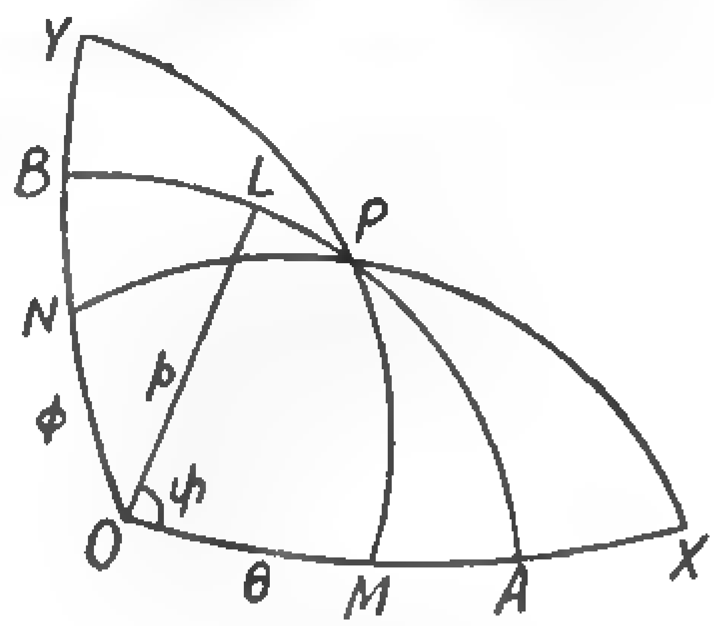


Fig. 63

Now from right-angled $\triangle$ AOL, we have

$$\cos\psi = \tan p . \cot OA$$

and from right-angled $\triangle$ BOL, we have.

$$\sin\psi = \tan p \cot OB.$$

Putting the values of $\cos\psi$ and $\sin\psi$ and cancelling $\tan p$, we get

$$\tan\theta \cot OA + \tan\phi \cot OB = 1. \quad \text{Hence proved.}$$

CHAPTER IV

SPHERICAL EXCESS

§ 1. Spherical Excess.

If A, B, C be the angles of a spherical triangle, then the quantity $A+B+C-\pi$ denoted by E is called the spherical excess of the triangle ABC.

$$\therefore \quad E=A+B+C-\pi=2S-\pi$$

or
$$S=\frac{E+\pi}{2}=\frac{E}{2}+\frac{\pi}{2}.$$

§ 2. To prove that

$$\sin\frac{E}{2}=\frac{\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}^{1/2}}{2\cos a/2 \cos b/2 \cos c/2}.$$

$$\sin\frac{a}{2}=\sqrt{\left\{\frac{-\cos S\cos(S-A)}{\sin B\sin C}\right\}}$$

and
$$\sin\frac{b}{2}=\sqrt{\left\{\frac{-\cos S\cos(S-B)}{\sin A\sin C}\right\}}$$

$$\therefore \quad \sin\frac{a}{2}.\sin\frac{b}{2}=\frac{-\cos S}{\sin C}\sqrt{\left\{\frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}\right\}}$$

$$=-\frac{\cos S}{\sin C}\cos\frac{c}{2} \quad (\text{§ 5·2 P. 22})$$

$$\therefore \quad \frac{\sin(a/2)\sin(b/2)\sin C}{\cos c/2}=-\cos S$$

$$=-\cos\left(\frac{E}{2}+\frac{\pi}{2}\right)=\sin\frac{E}{2} \qquad \ldots(1)$$

or
$$\frac{\sin(a/2)\sin b/2)}{\cos c/2}\cdot\frac{2n}{\sin a\sin b}=\sin\frac{E}{2}$$

or
$$\frac{\sin(a/2).\sin(b/2).2n}{\cos(c/2).2\sin(a/2)\cos(a/2)\,2\sin(b/2)\cos(b/2)}=\sin\frac{E}{2}$$

or $\sin \dfrac{E}{2}=\dfrac{\{\sin s.\sin (s-a) \sin (s-b) \sin (s-c)\}^{1/2}}{2 \cos (a/2) \cos (b/2) \cos (c/2)}$ ...(2)

$\because \quad n^2=\sin s \sin (s-a) \sin (s-b) \sin (s-c).$ (P. 21)

Alternative Method.

$$\sin \frac{E}{2}=\sin \left(\frac{A+B+C-\pi}{2}\right)=\sin \left(\frac{A+B}{2}-\frac{\pi-C}{2}\right)$$

$$=\sin \frac{A+B}{2} \sin \frac{C}{2}-\cos \frac{A+B}{2} \cos \frac{C}{2}.$$

Now

$$\frac{\sin \frac{A+B}{2}}{\cos C/2}=\frac{\cos \left(\frac{a-b}{2}\right)}{\cos c/2} \text{ and } \frac{\cos \frac{A+B}{2}}{\sin C/2}=\frac{\cos \left(\frac{a+b}{2}\right)}{\cos c/2}$$

(§ 10 P. 30)

$$\therefore \quad \sin \frac{E}{2}=\frac{\sin C/2.\cos C/2}{\cos c/2}\left\{\cos \left(\frac{a-b}{2}\right)-\cos \left(\frac{a+b}{2}\right)\right\}$$

$$=\frac{\sin C \sin (a/2) \sin b/2}{\cos c/2}=\frac{2n}{\sin a \sin b}\cdot\frac{\sin a/2.\sin b/2}{\cos c/2}$$

where $n^2=\sin s \sin (s-a) \sin (s-b) \sin (s-c).$ [P. 21]

Now change $\sin a$ and $\sin b$ into half and put for n^2 and we get the required result.

§ 3. To prove that

$$\tan \frac{E}{4}=\left\{\tan \frac{s}{2} \tan \left(\frac{s-a}{2}\right) \tan \left(\frac{s-b}{2}\right) \tan \left(\frac{s-c}{2}\right)\right\}^{1/2}$$

$$\tan \frac{E}{2}=\frac{\sin \frac{1}{4}(A+B+C-\pi)}{\cos \frac{1}{4}(A+B+C-\pi)}=\frac{\sin \left(\frac{A+B}{4}-\frac{\pi-C}{4}\right)}{\cos \left(\frac{A+B}{4}-\frac{\pi-C}{4}\right)}$$

$$=\frac{\sin \left(\frac{A+B}{2}\right)-\sin \left(\frac{\pi-C}{2}\right)}{\cos \left(\frac{A+B}{2}\right)+\cos \left(\frac{\pi-C}{2}\right)}$$

$$\therefore \quad \frac{\sin x-\sin y}{\cos x+\cos y}=\frac{2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}}{2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}}=\frac{\sin (x/2-y/2)}{\cos (x/2-y/2)}$$

$$=\frac{\cos\frac{C}{2}\left[\cos\frac{a-b}{2}-\cos\frac{c}{2}\right]}{\sin\frac{C}{2}\left[\cos\left(\frac{a+b}{2}\right)+\cos\frac{c}{2}\right]} \text{ by § 10 P. 30}$$

$$=\left\{\frac{\sin s\ \sin(s-c)}{\sin(s-a)\sin(s-b)}\right\}^{1/2}$$

$$\times\frac{2\sin\frac{a-b+c}{4}\sin\frac{c-a+b}{4}}{2\cos\frac{a+b+c}{4}\cos\frac{a+b-c}{4}}$$

$$=\left\{\frac{\sin s\ \sin(s-c)}{\sin(s-a)\sin(s-b)}\right\}^{1/2}\frac{\sin\left(\frac{s-b}{2}\right)\sin\left(\frac{s-a}{2}\right)}{\cos\frac{s}{2}.\cos\frac{s-c}{2}}$$

Now write $\sin s$ as $2\sin\frac{s}{2}\cos\frac{s}{2}$ etc.

$$\therefore\quad \tan\frac{E}{4}=\left\{\tan\frac{s}{2}\tan\left(\frac{s-a}{2}\right)\tan\left(\frac{s-b}{2}\right)\tan\left(\frac{s-c}{2}\right)\right\}^{1/2}$$

Alternative Method.

$$\frac{E}{2}=\frac{A+B+C-\pi}{2},\quad\therefore\quad \frac{A+B}{2}=\frac{\pi}{2}-\left(\frac{C-E}{2}\right).$$

$$\therefore\quad \sin\frac{A+B}{2}=\cos\frac{C-E}{2} \text{ and } \cos\frac{A+B}{2}=\sin\frac{C-E}{2}.$$

Putting the values of $\sin\frac{A+B}{2}$ and $\cos\frac{A+B}{2}$ in De Alembert's Analogies given in § 10 P. 30,

$$\frac{\cos\frac{C-E}{2}}{\cos C/2}=\frac{\cos\left(\frac{a-b}{2}\right)}{\cos c/2} \text{ and } \frac{\sin\frac{C-E}{2}}{\sin C/2}=\frac{\cos\left(\frac{a+b}{2}\right)}{\cos c/2}.$$

Applying componendo and dividendo and simplifying, we get

$$\tan \frac{E}{4} \tan \frac{2C-E}{4} = \tan \frac{s-a}{2} \tan \frac{s-b}{2}$$

and $$\tan \frac{E}{4} \cot \frac{2C-E}{4} = \tan s/2 . \tan \frac{s-c}{2}.$$

Multiplying the above results and taking square root, we get the value of $\tan (E/4)$ as given. Again dividing the above results and taking square root, we get

$$\tan \frac{2C-E}{4} = \left\{ \tan \frac{s-a}{2} \tan \frac{s-b}{2} \cot s/2 \cot \frac{s-c}{2} \right\}^{1/2}.$$

§ 4. To prove that

$$\cos \frac{E}{2} = \frac{1+\cos a+\cos b+\cos c}{4 \cos a/2 \cos b/2 \cos c/2}$$

$$= \frac{\cos^2 a/2+\cos^2 b/2+\cos^2 c/2-1}{2 \cos a/2 \cos b/2 \cos c/2}$$

$$\cos \frac{E}{2} = \cos \frac{A+B+C-\pi}{2} = \cos \left(\frac{A+B}{2} - \frac{\pi - C}{2} \right)$$

$$= \cos \frac{A+B}{2} \sin \frac{C}{2} + \sin \frac{A+B}{2} \cos \frac{C}{2}.$$

Now using De Alembert's analogies of § 10 P. 30, we get

$$\cos \frac{E}{2} = \frac{1}{\cos c/2} \left[\cos \frac{a+b}{2} \sin^2 \frac{C}{2} + \cos \frac{a-b}{2} \cos^2 \frac{C}{2} \right]$$

$$= \frac{1}{\cos c/2} \left[\cos \frac{a}{2} \cos \frac{b}{2} \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) \right.$$

$$\left. + \sin \frac{a}{2} \sin \frac{b}{2} \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right) \right]$$

$$= \frac{1}{\cos c/2} \left[\cos \frac{a}{2} \cos \frac{b}{2} + \sin \frac{a}{2} \sin \frac{b}{2} \cos C \right] \quad \ldots \quad \ldots (1)$$

Multiplying above and below by $4 \cos a/2 \cos b/2$, we get

$$\cos \frac{E}{2} = \frac{1}{4 \cos a/2 \cos b/2 \cos c/2}$$

$$\times [(1+\cos a)(1+\cos b)+\sin a \sin b \cos C],$$

$$\cos\frac{E}{2}=\frac{1}{4\cos a/2\cos b/2\cos c/2}$$
$$\times[1+\cos a+\cos b+(\cos a\cos b$$
$$+\sin a\sin b\cos C)],$$

$$\cos\frac{E}{2}=\frac{1}{4\cos a/2\cos b/2\cos c/2}[1+\cos a+\cos b+\cos c], \quad \ldots(2)$$

Changing $\cos a$ into $2\cos^2 (a/2)-1$, etc, we get the second form.

Again we have proved in § 2 result 2, P. 107 that

$$\sin\frac{E}{2}=\frac{n}{2\cos a/2\cos b/2\cos c/2}$$

and dividing by (2), we get

$$\tan\frac{E}{2}=\frac{2n}{1+\cos a+\cos b+\cos c}. \quad \ldots(3)$$

Another form for tan E/2.

We have already proved in § 2 Result 1 P. 96 that

$$\sin\frac{E}{2}=\frac{\sin a/2\sin b/2}{\cos c/2}\sin C. \quad \ldots(4)$$

Dividing (1) and (4), we get

$$\tan\frac{E}{2}=\frac{\sin a/2\sin b/2\sin C}{\cos a/2\cos b/2+\sin a/2\sin b/2\cos C} \quad \ldots(5)$$

§ 5. To find the values of

$\sin E/4$, $\cos E/4$ and $\tan E/4$.

We have proved in § 4 that

$$\cos\frac{E}{2}=\frac{1+\cos a+\cos b+\cos c}{4\cos a/2\cos b/2\cos c/2}$$

$$=\frac{\cos^2 a/2+\cos^2 b/2+\cos^2 c/2-1}{2\cos a/2\cos b/2\cos c/2}.$$

Now

$$\sin^2\frac{E}{4}=\tfrac{1}{2}(1-\cos E/2)$$

$$=\frac{2\cos a/2\cos b/2\cos c/2-\cos^2 a/2-\cos^2 b/2-\cos^2 c/2+1}{4\cos a/2\cos b/2\cos c/2}$$

$$=\frac{\sin\frac{s}{2}\sin\left(\frac{s-a}{2}\right)\sin\left(\frac{s-b}{2}\right)\sin\left(\frac{s-c}{2}\right)}{\cos a/2\cos b/2\cos c/2}$$

Similarly

$$\cos^2\frac{E}{4}=\tfrac{1}{2}(1+\cos E)$$

$$=\frac{2\cos a/2\cos b/2\cos c/2+\cos^2 a/2+\cos^2 b/2+\cos^2 c/2-1}{4\cos a/2\cos b/2\cos c/2}$$

$$=\frac{\cos\frac{s}{2}\cos\left(\frac{s-a}{2}\right)\cos\left(\frac{s-b}{2}\right)\cos\left(\frac{s-c}{2}\right)}{\cos a/2\cos b/2\cos c/2}$$

$$\therefore\quad \tan^2\frac{E}{4}=\tan\frac{s}{2}\tan\left(\frac{s-a}{2}\right)\tan\left(\frac{s-b}{2}\right)\tan\left(\frac{s-c}{2}\right).$$

Exercise

1. *If the angles of a spherical triangle be together equal to four right angles, show that*

$$\cos^2\frac{a}{2}+\cos^2\frac{b}{2}+\cos^2\frac{c}{2}=1.$$

[Alld. 50, Lucknow 56]

$A+B+C=2\pi$, $\therefore$ $E=A+B+C-\pi=\pi$; $\therefore$ $\frac{E}{2}=\frac{\pi}{2}$.

Now $\cos\frac{E}{2}=0$ and hence we get the result from § 4, P. 109.

Note. Independent proof of above is given at P. 70 Q. 23 (b).

2. *If* $a=b=\frac{\pi}{3}$ *and* $c=\frac{\pi}{2}$, *prove that* $E=\cos^{-1}\frac{7}{9}$.

[Raj. 58, Lucknow 56, Agra 48, 56]

$$\cos\frac{E}{2}=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}=\frac{2\sqrt{2}}{3} \qquad (\S\ 4\ \text{P. } 109)$$

$$\therefore\quad \cos E=2\cos^2\frac{E}{2}-1=2.\tfrac{8}{9}-1=\tfrac{7}{9}.$$

See also Q. 31, P. 79.

3. *If $\angle C=\frac{\pi}{2}$, find the values of $\sin\frac{E}{2}$ and $\cos\frac{E}{2}$.*

Putting $C=\frac{\pi}{2}$ in § 2 Result (1) and § 4 Result (1), we get

$$\sin\frac{E}{2}=\frac{\sin\frac{a}{2}\sin\frac{b}{2}}{\cos\frac{c}{2}} \text{ and } \cos\frac{E}{2}=\frac{\cos\frac{a}{2}\cos\frac{b}{2}}{\cos\frac{c}{2}}.$$

4. *If $a=b$ and $\angle C=\frac{\pi}{2}$, prove that $\tan E=\frac{1}{2}\frac{\sin^2 a}{\cos a}$.*

From Q. (3), we get $\tan\frac{E}{2}=\dfrac{\sin\frac{a}{2}\sin\frac{b}{2}}{\cos\frac{a}{2}\cos\frac{b}{2}}$.

$$\therefore\quad \tan E=\frac{2\tan\frac{E}{2}}{1-\tan^2\frac{E}{2}}=\frac{2\sin\frac{a}{2}\sin\frac{b}{2}\cos\frac{a}{2}\cos\frac{b}{2}}{\cos^2\frac{a}{2}\cos^2\frac{b}{2}-\sin^2\frac{a}{2}\sin^2\frac{b}{2}}$$

$$=\frac{\frac{1}{2}(\sin a\sin b)}{\frac{1}{4}[(1+\cos a)(1+\cos b)-(1-\cos a)(1-\cos b)]}$$

$$=\frac{2\sin a\sin b}{2(\cos a+\cos b)}=\frac{\sin^2 a}{2\cos a}\quad\because\quad a=b.$$

5. *If the angle C be a right angle, show that*

$$\frac{\sin^2 c}{\cos c}\cos E=\frac{\sin^2 a}{\cos a}+\frac{\sin^2 b}{\cos b}.$$

[Agra 50, Sagar 57, Allahabad 58, Rajputana 59]

We have to prove that

$$\cos E=\frac{\sin^2 a\cos b+\sin^2 b\cos a}{\cos a\cos b}\times\frac{\cos c}{\sin^2 c}$$

or $$\cos E=\frac{\cos b\,(1-\cos^2 a)+\cos a\,(1-\cos^2 b)}{\cos a\cos b}\cdot\frac{\cos a\cos b}{1-\cos^2 c}$$

$$\because\quad \angle C=\frac{\pi}{2},\quad \therefore\quad \cos c=\cos a\cos b.$$

$$=\frac{(\cos a+\cos b)-\cos a\cos b\,(\cos a+\cos b)}{1-\cos^2 c}$$

$$=\frac{(\cos a+\cos b)\,(1-\cos c)}{1-\cos^2 c}=\frac{\cos a+\cos b}{1+\cos c}$$

Now from Q. 3, $\cos E/2=\dfrac{\cos(a/2)\cos(b/2)}{\cos(c/2)}$ when $C=(\pi/2)$.

$$\therefore\quad \cos E=2\cos^2(E/2)-1=\frac{2\cos^2(a/2)\cos^2(b/2)-\cos^2(c/2)}{\cos^2(c/2)}$$

$$=\frac{(1+\cos a)\,(1+\cos b)-(1+\cos c)}{1+\cos c}$$

$$=\frac{\cos a+\cos b}{1+\cos c}.$$

Hence proved.

7. *A, B, C are the angular points of a spherical triangle, A′, B′, C′ are respectively the middle points of the opposite sides. If E be the spherical excess of the triangde, show that*

$$\cos\frac{E}{2}=\frac{\cos A'B'}{\cos(c/2)}=\frac{\cos B'C'}{\cos(a/2)}=\frac{\cos C'A'}{\cos(b/2)}.$$

(Agra 44, Rajputana 56)

$\cos A'B'=\cos(a/2)\cos(b/2)$
$\quad+\sin(a/2)\sin(b/2)\cos C$

$$\therefore\ \frac{\cos A'B'}{\cos(c/2)}=\frac{\cos(a/2)\cos(b/2)+\sin(a/2)\sin(b/2)\cos C}{\cos(c/2)}$$

$=\cos\dfrac{E}{2}.$ [by § 4 (1) P. 109]

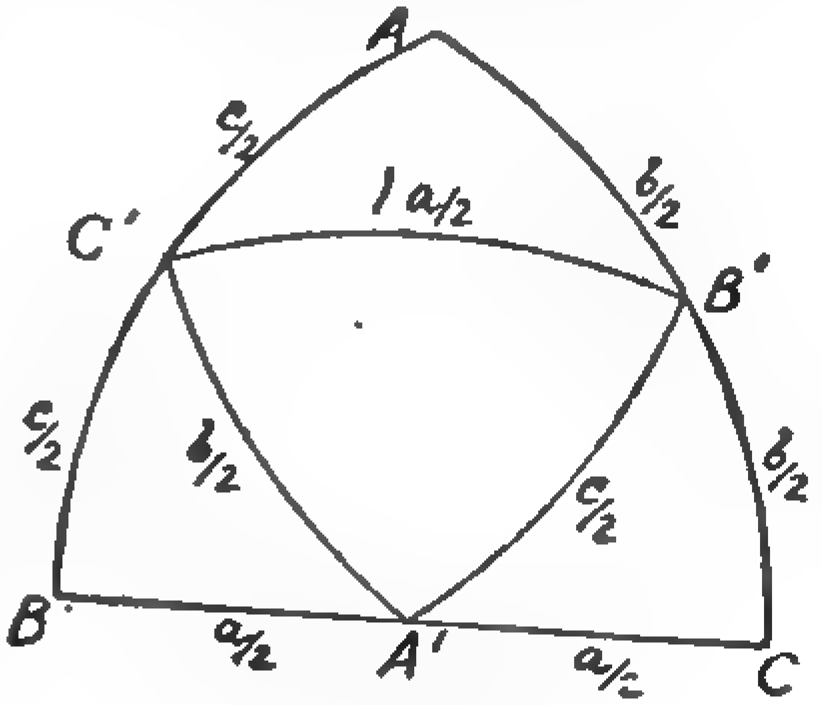


Fig. 64

The value of $\cos(E/2)$ as in

§ 4 can also be shown to be

$$\frac{\cos (b/2) \cos (c/2)+\sin (b/2) \sin (c/2) \cos A}{\cos (a/2)}=\frac{\cos B'C'}{\cos (a/2)} \text{ etc.}$$

Note. See Q. 5 (a) P. 51.

8. *Prove that the spherical excess of a triangle ABC is less than, equal to or greater than two right angles according as the expression $1+\cos a+\cos b+\cos c$ is greater than, equal to or less than zero.*

$$\cos E/2=\frac{1+\cos a \cos b+\cos c}{4 \cos (a/2) \cos (b/2) \cos (c/2)}.$$

(Jabalpur 58, Sagar 52)

Now the sides of a spherical triangle are each less than π; therefore $\cos (a/2)$, $\cos (b/2)$, $\cos (c/2)$ are all positive.

Hence if $1+\cos a+\cos b+\cos c$ is +ive, *i e.* > 0, then $\cos E/2 > 0$ or $E/2 < \pi/2$ *i.e.* $E < \pi$ or 2 right angles.

Similarly, if $1+\cos a+\cos b+\cos c$ is $\leqslant 0$; then then $\cos E/2 \leqslant 0$ or $E/2 \geqslant \pi/2$ or $E \geqslant \pi$ or 2 right angles.

9. *Prove that the sum of the angles of a right-angled triangle is less than four right angles.*

$$\cos \frac{E}{2}=\frac{\cos (a/2) \cos (b/2)}{\cos (c/2)} \text{ when } C=\pi/2,$$

(§ 4·1 P. 109)

or $\cos E/2=+$ive as is Q. 8, $\therefore$ $E/2 < \pi/2$ or $E < \pi$,

or $A+B+C-\pi < \pi$ or $A+B+C < 2\pi$ *i.e.* 4 right angles.

CHAPTER V

SMALL VARIATIONS

§ 1. Sometimes we are given that some elements of a spherical triangle undergo small changes by a known amount and some elements remain constant and we are required to find the corresponding changes in the other elements. For this we should write a relation between the elements which are either constant or which vary by known amount and the element whose variation is to be determined. The required variation is obtained by differentiation.

1. (a) *If the elements a, b, c, A of a spherical triangle ABC receive increments Δa, Δb, Δc and ΔA respectively, then show that*

$$\Delta \mathbf{a} = \mathbf{cos\ C}\ \Delta \mathbf{b} + \mathbf{cos\ B}\ \Delta \mathbf{c} + \mathbf{k\ sin\ b\ sin\ c}\ \Delta \mathbf{A}$$

where $k = \dfrac{\sin A}{\sin a}$ and two similar relations.

(Lucknow 1939)

The formula involving a b, c, A is cosine formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Differentiating, we get

$$-\sin a\ \Delta a = -\sin b\ \Delta b \cos c - \sin c\ \Delta c \cos b$$
$$+\cos b\ \Delta b \sin c \cos A + \cos c\ \Delta c \sin b \cos A$$
$$-\sin b \sin c \sin A\ \Delta A.$$

$$\therefore \quad \sin a\ \Delta a = (\sin b \cos c - \cos b \sin c \cos A)\ \Delta b$$
$$+(\sin c \cos b - \cos c \sin b \cos A)\ \Delta c$$
$$+\sin b \sin c \sin A\ \Delta A. \qquad \ldots(1)$$

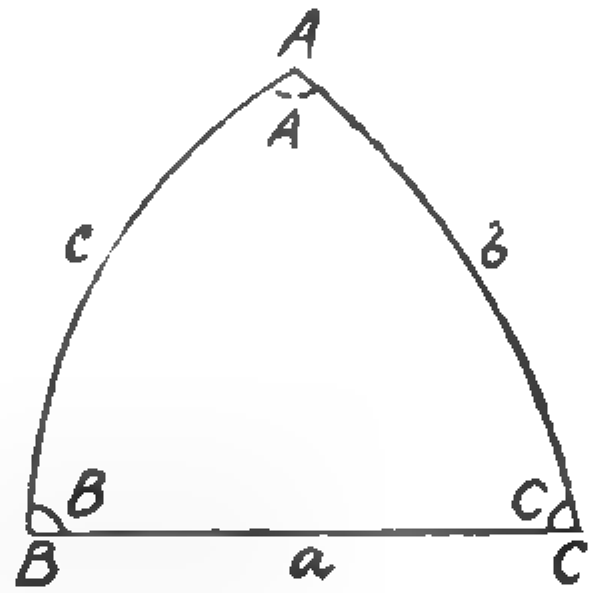


Fig. 65

Now by sine-cosine formula § 6, P. 23, we have

$\sin a \cos C = \sin b \cos c$
$\qquad - \cos b \sin c \cos A$

and $\sin a \cos B = \sin c \cos b$
$\qquad - \cos c \sin b \cos A.$

Hence from (1) we get

$\therefore \quad \sin a \,\Delta a = \sin a \cos C \,\Delta b$
$\qquad + \sin a \cos B \,\Delta c$
$\qquad + \sin b \sin c \sin A \,\Delta A.$

$\therefore \quad \Delta a = \cos C \,\Delta b + \cos B \,\Delta c + k \sin b \sin c \,\Delta A.$

$\because \quad \dfrac{\sin A}{\sin a} = k. \qquad \ldots(1)$

In a similar manner by writing the values of $\cos b$ and $\cos c$ and proceeding as above,

$\mathbf{\Delta b = \cos A \,\Delta c + \cos C \,\Delta a + k \sin c \sin a \,\Delta B,} \quad \ldots(2)$

$\mathbf{\Delta c = \cos B \,\Delta a + \cos A \,\Delta b + k \sin a \sin b \,\Delta C.} \quad \ldots(3)$

(b) *If a spherical triangle receives a small change which does not alter the sum of its three angles, show that the alteration in the lengths of the sides must satisfy the condition*

$$\Delta a \sin (S - A) + \Delta b \sin (S - B) + \Delta c \sin (S - C) = 0$$

where $2S = A + B + C.$ **(Agra 55, Rajputana 59)**

We are given that $A + B + C =$ constant.

$$\therefore \quad \Delta A + \Delta B + \Delta C = 0.$$

Now write down the values of Δa, Δb, Δc as in (1), (2) and (3) of part (a). Multiply (1) by $\sin A$ and (2) by $\sin B$ and (3) by $\sin C$ and keeping in view that

$\sin b \sin c \sin A = \sin c \sin a \sin B = \sin a \sin b \sin C,$

we get on adding the results thus obtained and putting $\Delta A + \Delta B + \Delta C = 0,$

$\Delta a \{\sin A - \sin B \cos C - \cos B \sin C\} + \Delta b \{\ \} + \Delta c \{\ \} = 0$

or $\Delta a \{\sin A - \sin (B + C)\} + \Delta b \{\ \} + \Delta c \{\ \} = 0$

or $\Delta a\left\{2\cos\frac{A+B+C}{2}\sin\frac{A-B-C}{2}\right\}+\Delta b\,\{\ \}+\Delta c\,\{\ \}=0.$

Cancel $-2\cos S$ and replace $\frac{B+C-A}{2}$ by $S-A$.

$\therefore\quad \Delta a\sin(S-A)+\Delta b\sin(S-B)+\Delta c\sin(S-C)=0.$

(c) *In a spherical triangle, if C and c remain constant while a and b receive the small increments Δa and Δb respectively, show that*

$$\frac{\Delta a}{\sqrt{(1-k^2\sin^2 a)}}+\frac{\Delta b}{\sqrt{(1-k^2\sin^2 b)}}=0 \text{ where } k=\frac{\sin A}{\sin a}.$$

(Lucknow 50)

Here we should write the relation involving a, b, c and C i. e. $\cos c=\cos a\cos b+\sin a\sin b\cos C$ and hence from part (a) result 3, we have

$$\Delta c=\cos B\,\Delta a+\cos A\,\Delta b+k\sin a\sin b\,\Delta C.$$

Since C and c are constants, $\therefore\ \Delta c=0$ and $\Delta C=0$.

$$\therefore\quad \cos B\,\Delta a+\cos A\,\Delta b=0,$$

or $$\frac{\Delta a}{\sqrt{(1-\sin^2 A)}}+\frac{\Delta b}{\sqrt{(1-\sin^2 B)}}=0$$

or $$\frac{\Delta a}{\sqrt{(1-k^2\sin^2 a)}}+\frac{\Delta b}{\sqrt{(1-k^2\sin^2 b)}}=0.$$

2. (a) *If the elements A, B, C, a of a spherical triangle ABC receive increments ΔA, ΔB, ΔC and Δa respectively, then show that*

$$\Delta A=-\cos c\,\Delta B-\cos b\,\Delta C+k^{-1}\sin B\sin C\,\Delta a,$$

where $k=\frac{\sin A}{\sin a}$ and two similar relations.

The formula involving A, B, C and a is supplemental cosine formula

$$\cos A=-\cos B\cos C+\sin B\sin C\cos a.$$

Differentiating, we get

$$-\sin A\,\Delta A=(\sin B\,\Delta B)\cos C+(\sin C\,\Delta C)\cos B$$
$$+\cos B\,\Delta B\sin C\cos a+\cos C\,\Delta C\sin B\cos a$$
$$-\sin B\sin C\sin a\,\Delta a$$

or $\quad -\sin A\,\Delta A=(\sin B\cos C+\cos B\sin C\cos a)\,\Delta B$
$$+(\sin C\cos B+\cos C\sin B\cos a)\,\Delta C$$
$$-\sin B\sin C\sin a\,\Delta a.$$

We have written in Q. 1 by using sine-cosine formula the relations

$$\sin a\cos C=\sin b\cos c-\cos b\sin c\cos A$$

and $\quad \sin a\cos B=\sin c\cos b-\cos c\sin b\cos A.$

By using the property of polar triangles, *i. e.* replacing a by $\pi-A$ etc. and A by $\pi-a$ etc., we get

$$-\sin A\cos c=-\sin B\cos C-\cos B\sin C\cos a$$

and $\quad -\sin A\cos b=-\sin C\cos B-\cos C\sin B\cos a.$

$\therefore \quad -\sin A\,\Delta A=(\sin A\cos c)\,\Delta B+(\sin A\cos b)\,\Delta C$
$$-\sin B\sin C\sin a\,\Delta a$$

or $\quad \Delta A=-\cos c\,\Delta B-\cos b\,\Delta C+\sin B.\sin C.(1/k).\Delta a,$

$$\because \quad \frac{\sin A}{\sin a}=k.$$

In a similar manner by writing the values of $\cos B$ and $\cos C$ and proceeding as above,

$$\Delta B=-\cos a\,\Delta C-\cos c\,\Delta A+\sin C\sin A.(1/k).\Delta b,$$
$$\Delta C=-\cos b\,\Delta A-\cos a\,\Delta B+\sin A\sin B.(1/k)\,\Delta c.$$

Note. The above formula could directly be obtained from the corresponding formula of Q. 1 (a) by replacing a by $\pi-A$, A by $\pi-a$ and by replacing Δa by $-\Delta A$ and ΔA by $-\Delta a$ and so on.

and $$k=\frac{\sin A}{\sin a}=\frac{\sin(\pi-a)}{\sin(\pi-A)}=\frac{\sin a}{\sin A}=\frac{1}{k}.$$

(b) *Under what conditions can a spherical triangle undergo a small change such that $\Delta a=-\Delta b=\Delta A=\Delta B=0$ while Δc and ΔC are not zero ?*

From Q. 1, putting

$$\Delta a=\cos C\,\Delta b+\cos B\,\Delta c+k\sin b\sin c\,\Delta A,$$
$$0=0+\cos B\,\Delta c+0 \quad \text{[by given condition]}.$$

Now $\quad \Delta c\neq 0; \quad \therefore \quad \cos B=0, \quad i.\,e. \quad B=\pi/2.$

Similarly writing for Δb, we get $A=\pi/2$.

Again from Q. 2 by writing the values of ΔA and ΔB and proceeding as above we shall have

$$a=\pi/2 \quad \text{and} \quad b=\pi/2.$$

3. *In a spherical triangle if C and c remain constant, determine the relations between the variations of any other pair of elements.*

We know that there are six elements in a triangle out of which C and c are constants. Hence the remaining two are to be chosen from $a, b,$ A and B. We can have ${}^4C_2$, *i.e.* 6 pairs out of these four elements. We are required to find the relations involving the small variations of these pairs. As already stated we shall write a relation between the elements which are constant and those whose small variations are to be found.

(i) Relation between variations of a and b when c and C are constants.

$$\cos c=\cos a \cos b+\sin a \sin b \cos C.$$

Proceeding as in Q. 1 but here keeping in view that c and C are constants, *i.e.* Δc and ΔC are zero, we get

$$\cos B\ \Delta a+\cos A\ \Delta b=0.$$

(ii) Relation between a and A when c and C are constants.

$$\frac{\sin a}{\sin A}=\frac{\sin c}{\sin C}.$$

Taking log and differentiating, we get

$$\frac{1}{\sin a}.\cos a\ \Delta a-\frac{1}{\sin A}\cos A\ \Delta A=0$$

or $$\cot a\ \Delta a=\cot A\ \Delta A.$$

(iii) Relation between b and B when c and C are constants.

$$\cot b\ \Delta b=\cot B\ \Delta B \text{ as in (ii).}$$

(iv) Relation between A and B when c and C are constants.

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Proceeding as in Q. 2 and putting $\Delta c = 0$ and $\Delta C = 0$, we get

$$\mathbf{\cos b \,\Delta A + \cos a \,\Delta B = 0.}$$

(v) Relation between a and B when c and C are constants.

Applying the formula of consecutive four choosing the four elements to be C, a, B and c, we get

$$\cos a . \cos B = \sin a \cot c - \sin B \cot C.$$

Differentiating, keeping in view that c and C are constants,

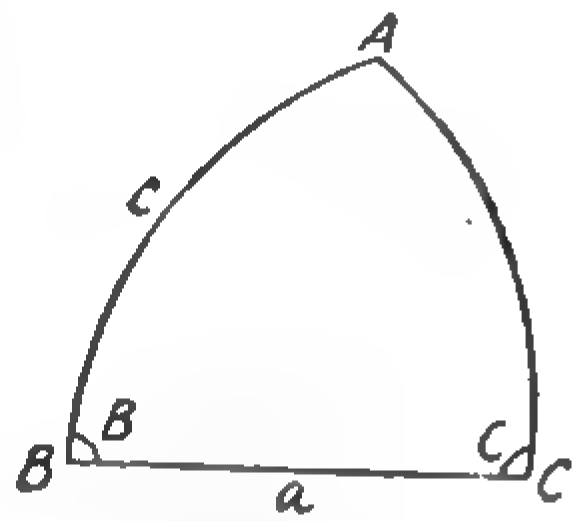


Fig. 66

$$-\sin a \,\Delta a \cos B - \sin B \,\Delta B \cos a$$
$$= \cos a \,\Delta a \cot c - \cos B \,\Delta B \cot C.$$

$$\therefore \quad (\sin a \cos B + \cos a \cot c)\,\Delta a$$
$$= (\cos B \cot C - \sin B \cos a)\,\Delta B$$

or $$\frac{\cos a \cos c + \sin a \sin c \cos B}{\sin c}\,\Delta a$$
$$= \frac{\cos B \cos C - \sin B \sin C \cos a}{\sin C}\,\Delta B$$

or $$\frac{\cos b}{\sin c}\,\Delta a = -\frac{\cos A}{\sin C}\,\Delta B$$

or $$\cos b \,\Delta a = -\frac{\sin c}{\sin C}\cos A \,\Delta B$$

or $$\cos b \,\Delta a = -\frac{\sin b}{\sin B}\cos A \,\Delta B \qquad \text{[sin form]}$$

or $$\mathbf{\cot b \sin B \,\Delta a + \cos A \,\Delta B = 0.}$$

(vi) Proceeding as above we can find **a relation between b and A.**

$$\mathbf{\cot a \sin A \,\Delta b + \cos B \,\Delta A = 0.}$$

4. *In a spherical triangle if A and C remain constant, prove the following relations connecting the small variations of other elements :*

(i) $\sin C\,\Delta b=\sin a\,\Delta B$, (ii) $\sin C\,\Delta b=-\tan a\,\Delta C$,
(iii) $\tan C\,\Delta a=\sin a\,\Delta B$, (iv) $\tan C\,\Delta a=-\tan a\,\Delta C$,
(v) $\cos C\,\Delta b=\Delta a$, (vi) $\cos a\,\Delta B=-\Delta C$.

(i) b and B, A and c.

Use consecutive four formula.

(ii) b, C. A and c.

Use consecutive four formula.

(iii) a, B, A and c.

Use consecutive four formula.

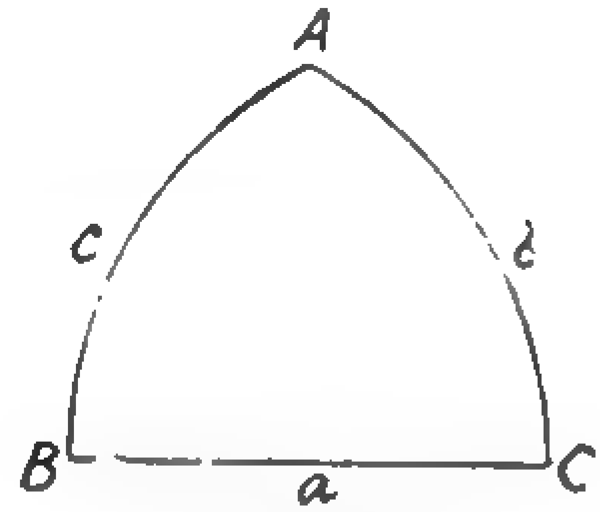


Fig. 67

(iv) a, C, A and c.

Use sine formula and take logarithmic differentiation.

(v) a, b and A, c.

Use cosine formula and proceed as in Q. 1 or putting $\Delta A=0$ and $\Delta c=0$ in Q. 1 we get the result directly.

(vi) B, C and A, c

Use supplemental cosine formula and proceed as in Q. 2 or putting $\Delta A=0$ and $\Delta c=0$ in Q. 2 we get the result.

5. *Supposing b and c to remain constant, prove the following equations connecting the small variations of pairs of the other elements :*

(i) $\tan C\,\Delta B=\tan B\,\Delta C$, (ii) $\cot C\,\Delta a+\sin a\,\Delta B=0$,
(iii) $\Delta a=\sin c\sin B\,\Delta A$, (iv) $\sin B\cos C\,\Delta A+\sin A\,\Delta B=0$.

6. *In a spherical triangle if B and C remain constant, prove the following relations connecting the small variations of other elements :*

(i) $\tan c\,\Delta b=\tan b\,\Delta c$, (ii) $\cot c\,\Delta A=\sin A\,\Delta b$,
(iii) $\Delta A=\sin b\sin C\,\Delta a$, (iv) $\sin B\cos c\,\Delta a=\sin A\,\Delta b$.

7. *In any spherical triangle, prove that*

$$\cot A\,\Delta A+\cot B\,\Delta B=\cot b\,\Delta b+\cot a\,\Delta a.$$

We know that $\sin A = k \sin a$.

Taking log and differentiating, we get

$$\frac{1}{\sin A} \cdot \cos A \, \Delta A = 0 + \frac{1}{\sin a} \cos a \, \Delta a$$

or
$$\cot A \, \Delta A = \cot a \, \Delta a.$$

Similarly $\cot B \, \Delta B = \cot b \, \Delta b$.

Adding the above we get the required result.

8. *In any spherical triangle, show that*

$$\sin a \, \Delta B = \sin C \, \Delta b - \sin B \cos a \, \Delta c - \sin b \cos C \, \Delta A.$$

The formula involving B, c, A, b is that of consecutive four.

$$\cos c \cos A = \sin c \cot b - \sin A \cot B.$$

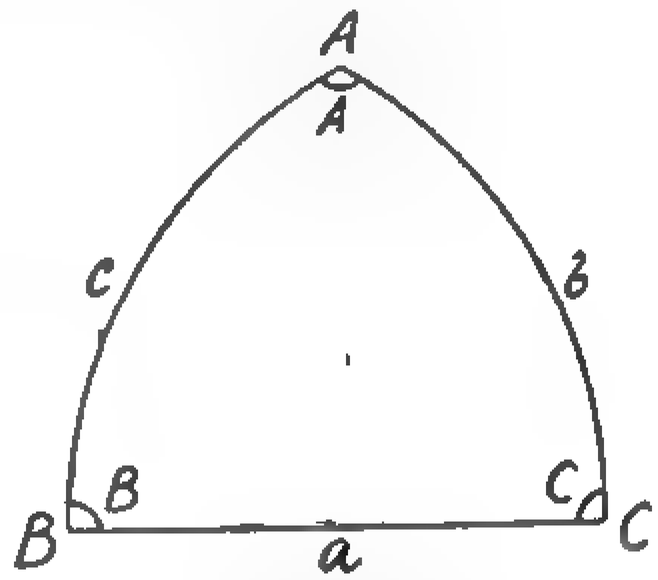


Fig. 68

Differentiating,

$$-\sin c \, \Delta c \cos A - \sin A \, \Delta A \cos c$$
$$= \cos c \, \Delta c \cot b - \sin c \operatorname{cosec}^2 b \, \Delta b$$
$$- \cos A \, \Delta A \cot B$$
$$+ \sin A \operatorname{cosec}^2 B \, \Delta B.$$

$$\frac{\sin A}{\sin^2 B} \Delta B = \frac{\sin c}{\sin^2 b} \Delta b - \frac{\cos c \cos b + \sin b \sin c \cos A}{\sin b} \Delta c$$
$$+ \frac{\cos A \cos B - \sin A \sin B \cos c}{\sin B} \Delta A$$

$$\frac{k \sin a}{k^2 \sin^2 b} \Delta B = \frac{\sin c}{\sin^2 b} \Delta b - \frac{\cos a}{\sin b} \Delta c - \frac{\cos C}{k \sin b} \Delta A.$$

Multiply by $k \sin^2 b$.

$$\sin a \, \Delta B = k \sin c \, \Delta b - k \sin b \cos a \, \Delta c - \sin b \cos C \, \Delta A.$$

Replace $k \sin c$ by $\sin C$ and $k \sin b$ by $\sin B$.

$\therefore \quad \sin a \, \Delta B = \sin C \, \Delta b - \sin B \cos a \, \Delta c - \sin b \cos C \, \Delta A.$

9. *In determining H (hour angle) a sailor makes error Δz in z (zenith distance) and $\Delta \phi$ in ϕ (latitude); show that the error in H is given by*

$$\Delta H = \cot A \sec \phi \, \Delta \phi + \sec \phi \operatorname{cosec} A \, \Delta z.$$

$\cos z = \sin\phi\sin\delta$

$+\cos\phi\cos\delta\cos H.$

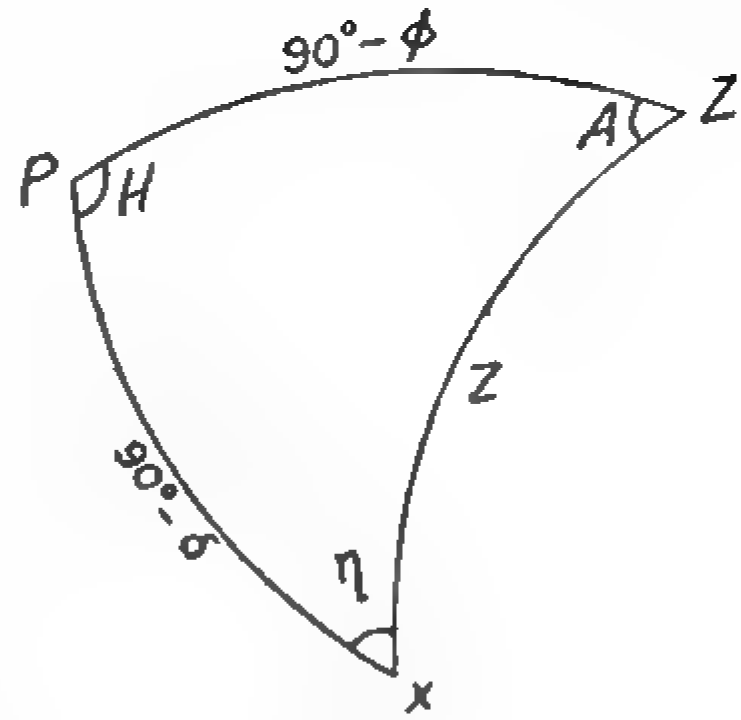


Fig. 69

Taking differentials, δ being constant,

$-\sin z\,\Delta z = \cos\phi\,\Delta\phi\,\sin\delta$

$-\sin\phi\,\Delta\phi\cos\delta\cos H$

$-\cos\phi\cos\delta\sin H\,\Delta H.$

$\therefore \cos\phi\cos\delta\sin H\,\Delta H$

$=(\cos\phi\sin\delta-\sin\phi\cos\delta\cos H)\,\Delta\phi$

$+\sin z\,\Delta z$

$=\sin z\cos A\,\Delta\phi+\sin z\,\Delta z$ by sine-cosine formula.

$$\Delta H=\frac{\sin z\cos A}{\cos\phi\cos\delta\sin H}\Delta\phi+\frac{\sin z}{\cos\phi\cos\delta\sin H}\Delta z.$$

Now by sine formula, $\dfrac{\sin z}{\sin H}=\dfrac{\cos\delta}{\sin A}$.

Putting in above,

$$\Delta H=\frac{\cos A}{\cos\phi}\cdot\frac{1}{\sin A}\Delta\phi+\frac{1}{\cos\phi}\cdot\frac{1}{\sin A}\Delta z$$

or $$\Delta H=\sec\phi\cot A\,\Delta\phi+\sec\phi\,\mathrm{cosec}\,A\,\Delta z.$$

10. *If in the spherical triangle PZX of Q. 9, ϕ and δ be constant prove that $\Delta\eta=-\cos\phi\cos A\,\mathrm{cosec}\,z\,\Delta H$.*

The relation involving ϕ, H, δ, η is of consecutive four.

$\cos(90-\delta)\cos H=\sin(90-\delta)\cot(90-\phi)-\sin H\cot\eta$

or $\sin\delta\cos H=\cos\delta\tan\phi-\sin H\cot\eta$

Differentiating and keeping ϕ, δ constant,

$-\sin\delta\sin H\,\Delta H=-\cos H\,\Delta H\cot\eta+\sin H\,\mathrm{cosec}^2\,\eta\,\Delta\eta.$

$$\frac{\sin H}{\sin^2\eta}\Delta\eta=-\left(\frac{-\cos H\cos\eta+\sin H\sin\eta\sin\delta}{\sin\eta}\right)\Delta H.$$

$\therefore\ \Delta\eta=\dfrac{\sin\eta}{\sin H}\cdot(-\cos A)\,\Delta H$ by supplemental cosine formula

$=-\dfrac{\sin(90-\phi)}{\sin z}\cos A\,\Delta H$ by sine formula

$=-\cos\phi\cos A\,\mathrm{cosec}\,z\cdot\Delta H.$

11. *In the spherical triangle PZX of Q. 9, ϕ and δ are constants ; prove that $\Delta A = -(\sin\phi - \cos\phi \cot z \cos A)\,\Delta H$.*

Here we should write down the relation of A, ϕ, H and δ.

$$\cos(90-\phi)\cos H = \sin(90-\phi)\cot(90-\delta) - \sin H \cot A.$$

$$\sin\phi\cos H = \cos\phi\tan\delta - \sin H\cot A.$$

$$-\sin\phi\sin H\,\Delta H = -\cos H\,\Delta H\cot A + \sin H\operatorname{cosec}^2 A\,\Delta A;$$

$$\therefore\quad \frac{\sin H}{\sin^2 A}\Delta A = \frac{\cos H\cos A - \sin H\sin A\sin\phi}{\sin A}\,\Delta H.$$

$$\Delta A = \frac{\sin A}{\sin H}(-\cos\eta).\Delta H$$ by supplemental cosine formula

$$= -\frac{\sin(90-\delta)}{\sin z}\cos\eta\,\Delta H = -\frac{\cos\delta\cos\eta}{\sin z}\Delta H$$

Now by sine-cosine formula, we have

$$\sin(90-\delta)\cos\eta = \sin z\cos(90-\phi) - \cos z\sin(90-\phi)\cos A$$

or $$\cos\delta\cos\eta = \sin z\sin\phi - \cos z\cos\phi\cos A;$$

$$\therefore\quad \Delta A = -\frac{\sin z\sin\phi - \cos z\cos\phi\cos A}{\sin z}\Delta H$$

$$= -(\sin\phi - \cot z\cos\phi\cos A)\,\Delta H.$$

12. *By refraction z is decreased by an amount $k\tan z$, and ϕ, A remain constant ; prove that*

(i) $\Delta\delta = K\tan z\cos\eta$, (ii) $\Delta H = -k\tan z\sin\eta\sec\delta$,
(iii) $\Delta\eta = k\tan z\tan\delta\sin\eta$

We are given that $\Delta z = -k\tan z$ as z is decreased. ϕ and A are constants and we are to find $\Delta\delta$.

$$\cos(90-\delta) = \cos z\cos(90-\phi) + \sin z\sin(90-\phi)\cos A$$

or $$\sin\delta = \cos z\sin\phi + \sin z\cos\phi\cos A.$$

$$\therefore\quad \cos\delta\,\Delta\delta = -\sin z\,\Delta z\sin\phi + \cos z\,\Delta z\cos\phi\cos A$$

$$= -(\sin z\sin\phi - \cos z\cos\phi\cos A)\,\Delta z$$

$$= k\tan z\,\sin(90-\delta)\cos\eta\,.$$

By sine-cosine formula and $\because\ \Delta z = -K\tan z$ (given),

$$\therefore\quad \Delta\delta = k\tan z\cos\eta.$$

Similarly we can prove other parts.

CHAPTER VI

MISCELLANEOUS PROBLEMS

1. (a)

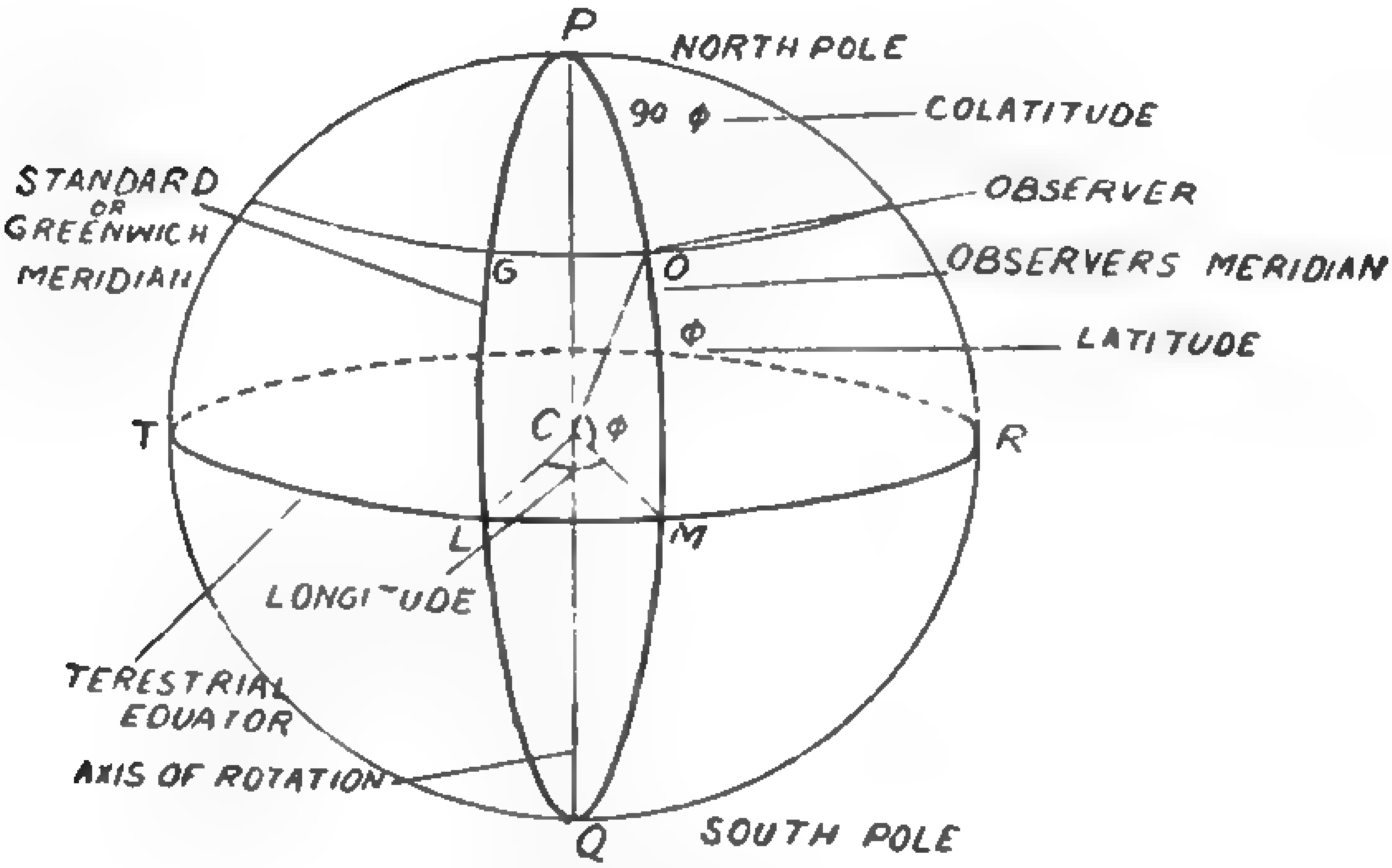


Fig. 69

The above figure represents earth supposed to be a sphere rotating about its axis through its centre from West to East. The extremities of the axis P and Q are called **north and south pole** respectively. The great circle TR whose plane is perpendicular to the axis PQ, *i.e.* whose poles are P and Q is called the **terrestrial equator.** Secondaries to the equator *i.e.* great circles passing through P and Q are called **Meridians.** If O be an observer on the earth's surface then the great circle POQ is called **Observer's Meridian.** The particular meridian which passes through Greenwich is called **Standard Meridian.**

Position on the earth. Let O be the position of the observer and PGQ be the prime meridian meeting the

equator at L and POQ be the observer's meridian cutting the equator in M.

Longitude. The longitude of O is defined to be the arc of the equator intercepted between the prime meridian and the meridian through O. Thus arc LM=longitude. It can also be measured by the plane angle LCM or spherical angle LPM *i.e* angle between two great circles. It is to be noted here that all places on the same meridian have the same longitude.

Longitudes are measured from 0 to 180° East of Greenwich meridian. When the place is east of Greenwich it is called east longitude. Similarly when the place is west of Greenwich, it has longitude west and is measured from 0 to 180° west of Greenwhich meridian.

Latitude. The arc of the observer's meridian intercepted between the equator and the observer is called the latitude of the observer. Thus arc OM or $\angle MCO$ is called latitude of O and is generally denoted by ϕ. If the observer be between the equator and north pole, the latitude is said to be north and is measured from 0 to 90° from the equator. Similarly if the place be between the equator and south pole then its latitude is south and is measured from 0 to 90° from the equator.

Colatitude. The arc of the observer's meridian between the north pole and the observer is called Colatitude. Thus PO is the colatitude. Again since $PM=90°$ and $OM=\phi$,

$\therefore$ Colatitude $PO=90-\phi$.

Parallel of latitude. All places which have the same latitude will lie on a small circle parallel to the equator and it is called parallel of latitude.

Exercise

1. *Two ports are in the same parallel of latitude, their common latitude being l and their difference of longitude 2λ ; show that the*

saving of distance in sailing from one to the other on the great circle instead of sailing due east or west is

$$2r\,\{\lambda \cos l - \sin^{-1}(\sin\lambda \cos l)\},$$

λ being expressed in circular measure and r being the radius of the earth. **(Utkal 55; Agra 54, Rajputana 58, Allahabad 57, 60, Nagpur 61, Sagar 1952)**

TR represents the equator and A and B be the two ports on the same parallel of latitude. P is the pole of equator and the difference of longitudes between A and $B = 2\lambda$ so that $\angle APB = 2\lambda$

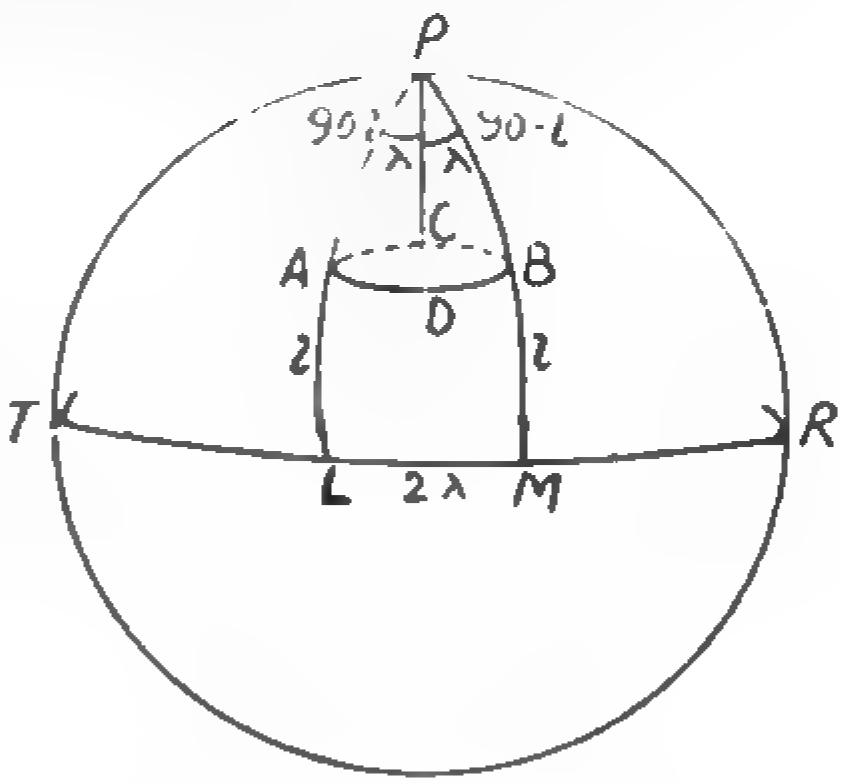


Fig. 70

i.e. arc $LM = 2\lambda$ radians

$= 2\lambda r$ in linear units...(1)

$\therefore$ $l/r = \theta$ radian.

Also $AL = l = BM.$

$\therefore$ $PA = 90 - l = PB.$ ...(2)

Now sailing due east or west means that the ship sails along the small circle arc ADB, where small circle arc

$ADB =$ parallel great circle arc $LM.\cos l$

$= 2\lambda r.\cos l$ linear units. ...(3)

Join AB by a great circle arc and let PC be perpendicular to it; then clearly C is the middle point of great circle arc ACB as $\Delta\,PAB$ is isosceles Now choosing AC as middle in the right-angled triangle ACP, we get

$$\sin AC = \cos(90-\lambda)\cos(90-\underline{90-l}) = \sin\lambda\cos l.$$

$\therefore$ $AC = \sin^{-1}(\sin\lambda \cos l).$

$\therefore$ great circle arc $AB = 2AC$

$= 2\sin^{-1}(\sin\lambda\cos l)$ radians

$= 2r\sin^{-1}(\sin\lambda\cos l)$ in linear units. ...(4)

Thus the saving in distance

$=$ Arc $ADB -$ Arc ACB

$= 2r\,\{\lambda\cos l - \sin^{-1}(\sin\lambda\cos l)\}$ [by (3) and (4)] Proved.

2. *Show that in sailing from one meridian to a place in the same latitude on another meridian the distance saved by sailing along a great circle instead of sailing due east and west is maximum for latitude* $\cos^{-1}\sqrt{(\operatorname{cosec}^2\lambda-1/\lambda^2)}$ *where* 2λ *is the difference of longitude of the two meridians.* **(Agra 51, 55, 58, Rajputana 55)**

Let the latitude be l, so that by Q. 1 the saving S is given by

$$s=2r\,\{\lambda\cos l-\sin^{-1}(\sin\lambda\cos l)\}.$$

We are to find the value of l corresponding to which S is maximum. Now λ and r being constants, therefore S is a function of l and for maximum value of S, we must have

$$\frac{ds}{dl}=0.$$

$$\frac{ds}{dl}=2r\left\{-\lambda\sin l-\frac{1}{\sqrt{(1-\sin^2\lambda\cos^2 l)}}(-\sin\lambda\sin l)\right\}=0$$

or $$2r\sin l\left\{\frac{\sin\lambda}{\sqrt{(1-\sin^2\lambda\cos^2 l)}}-\lambda\right\}=0.$$

$r\neq 0$. If $\sin l=0$, then $l=0$, *i.e.* the two places are on the equator and as such there arises no question of saving because on the equator we travel along the great circle. Hence we must have

$$\frac{\sin\lambda}{\sqrt{(1-\sin^2\lambda\cos^2 l)}}-\lambda=0$$

or $\sin^2\lambda=\lambda^2-\lambda^2\sin^2\lambda\cos^2 l$ or $\cos^2 l=\dfrac{\lambda^2-\sin^2\lambda}{\lambda^2\sin^2\lambda}$

or $$\cos l=\sqrt{\left(\operatorname{cosec}^2\lambda-\frac{1}{\lambda^2}\right)}.$$

$$\therefore\quad l=\cos^{-1}\sqrt{\left(\operatorname{cosec}^2\lambda-\frac{1}{\lambda^2}\right)}.$$

3. *A and B are two places on the earth's surface with same latitude ϕ, the difference of longitude is $2l$. Prove (i) the highest latitude reached by the great circle AB is $\tan^{-1}(\tan\phi \sec l)$. (ii) The distance measured along the parallel of latitude between A and B exceeds the great circle distance AB by*

$$2 \operatorname{cosec}^{-1} \{(l \cos\phi - \sin^{-1}(\sin l \cos\phi)\}.$$

From P draw PC perpendicular to great circle AB which is therefore least and hence the latitude of C i. e. its distance from equator is greatest. By symmetry C is mid. point of AB. If λ be the latitude of C then $PC=90-\lambda$.

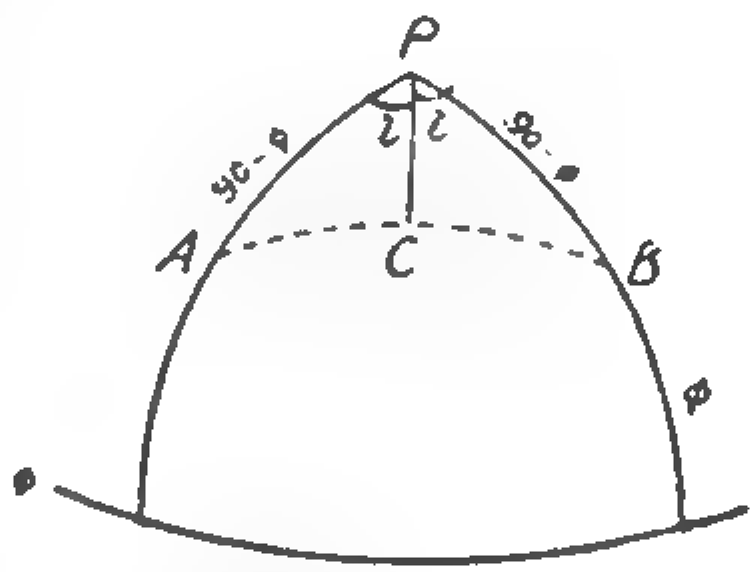


Fig. 71

Now choosing $\angle APC$ as middle in right angled triangle APC, we get

$$\sin(90-l)=\tan(90-\underline{90-\phi})\tan(90-\lambda)$$

or $$\cos l=\tan\phi.\cot\lambda, \quad \therefore \quad \tan\lambda=\tan\phi \sec l.$$

$$\therefore \quad \lambda=\tan^{-1}(\tan\phi \sec l).$$

2nd part follows from Q. 1.

4. *Show that the greatest distance that could be saved in a single voyage by sailing along a great circle instead of a parallel of latitude is $a[2\sin^{-1}(2/\pi)+\sqrt{(\pi^2-4)}-\pi]$ where a is the radius of the earth.*

In this case the two ports should be so situated that the difference of their logitude is π radians, i. e. the two places should be on the same meridian. If l be the latitude, then the small circle between them is

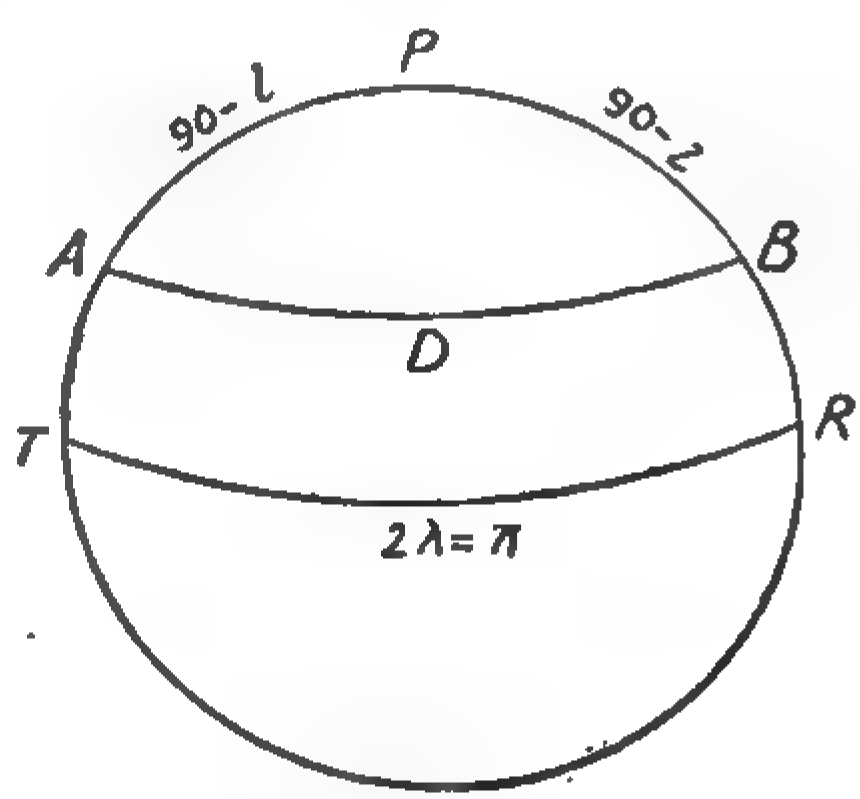


Fig. 72

Arc ADB=great circle $TR \times \cos l$

$=\pi \cos l$ radians

$=\pi a \cos l$ in linear units. ...(1)

Also great circle distance between them is arc APB

$=AP+PB=\pi-2l$ in radians$=(\pi-2l)\,a$ in linear units.

$\therefore$ saving $S=a\,(\pi \cos l-\pi+2l)$.

For max. value of S, we must have $dS/dl=0$.

$\therefore\ dS/dl=a\,(-\pi \sin l+2)$, $\therefore\ \sin l=2/\pi$.

$\therefore\ S=a\,[\pi\sqrt{(1-\sin^2 l)}-\pi+2l]$

$=a\,[\pi\sqrt{(1-4/\pi^2)}-\pi+2\sin^{-1} 2/\pi]$

$=a\,[\sqrt{(\pi^2-4)}-\pi+2\sin^{-1} 2/\pi]$. Proved.

5. *If a ship be sailing uniformly along a great circle and the observed latitudes be l_1, l_2, l_3 at equal intervals of time in each of which the distance traversed is s, show that*

$$s=r\cos^{-1}\frac{\sin\dfrac{l_1+l_3}{2}\cos\dfrac{l_1-l_3}{2}}{\sin l_2}$$

where r denotes the earth's radius. Show also that the change of longitude may also be found in terms of the three latitudes.

(Agra 53, 55, Alld. 59, Benares 56, Raj. 59, Pb. 49)

Let TR be the equator P its pole. A, B, C are three points on a great circle whose latitudes are l_1, l_2, l_3, so that $PA=90-l_1$, $PB=90-l_2$ and $PC=90-l_3$. Also $AB=BC=s$ in linear measure$=s/r$ in radians.

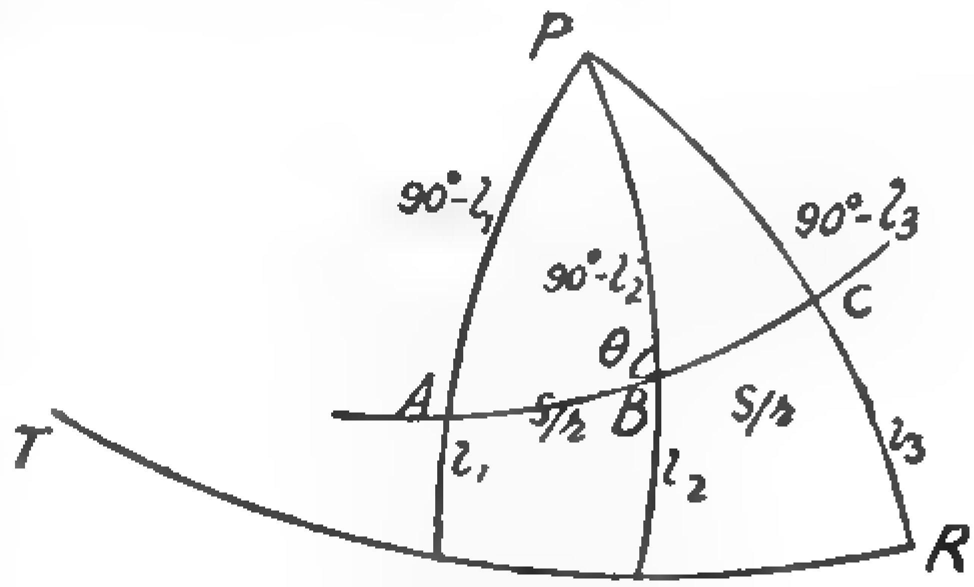


Fig. 73

If $\angle PBA=\theta$, then $\angle PBC=\pi-\theta$.

$\therefore\ \cos(90-l_1)=\cos(s/r)\cos(90-l_2)$

$+\sin(s/r)\sin(90-l_2)\cos\theta$. ...(1)

and $\cos(90-l_3)=\cos(s/r)\cos(90-l_2)$

$$+\sin(s/r)\sin(90-l_2)\cos(\pi-\theta)\ldots(2)$$

Adding (1) and (2), thereby eliminating θ, we get

$$\sin l_1+\sin l_3=2\cos(s/r)\sin l_2$$

or
$$2\sin\frac{l_1+l_3}{2}\cos\frac{l_1-l_3}{2}=2\cos\frac{s}{r}\sin l_2;$$

$$\therefore\quad s=r\cos^{-1}\left\{\frac{\sin\frac{l_1+l_3}{2}\cos\frac{l_1-l_3}{2}}{\sin l_2}\right\}.$$ Hence proved.

Again change in longitude is given by $\angle APC$

$$\cos APC=\frac{\cos\frac{2s}{r}-\cos(90-l_1)\cos(90-l_3)}{\sin(90-l_1)\sin(90-l_3)}$$

$$=\frac{\cos\frac{2s}{r}-\sin l_1\sin l_3}{\cos l_1\cos l_3}$$

Now we can put the values of $\cos(2s/r)$ in terms of $\cos(s/r)$ *i.e.* in terms of l_1, l_2, l_3 from the result proved above. Hence change in longitude *i.e.* $\angle APC$ can be found in terms of l_1, l_2 and l_3.

6. *Show that if L be the length of the arc of a great circle on the earth (supposed to be a sphere of radius R) extending from latitude λ, longitude l_1 to latitude λ_2 and longitude l_2, then*

$$L=R\cos^{-1}(\sin\lambda_1\sin\lambda_2\sec^2\phi)$$

when $\tan^2\phi=\cot\lambda_1\cot\lambda_2\cos(l_1-l_2)$ and that the highest latitude reached by the great circle will be

$$\cos^{-1}\left\{\cos\lambda_1\cos\lambda_2\sin(l_1\sim l_2)\operatorname{cosec}\frac{L}{R}\right\}.$$

(Sagar 56, 58, Agra 44)

$Q_1Q_2 = L$ in linear units

$= \frac{L}{R}$ in radians.

$\therefore \cos \frac{L}{R} = \sin \lambda_1 \sin \lambda_2$

$+ \cos \lambda_1 \cos \lambda_2$

$\times \cos.(l_1 - l_2)$

$= \sin \lambda_1 \sin \lambda_2$

$\times \{1 + \cot \lambda_1 \cot \lambda_2$

$\times \cos (l_1 - l_2)\}.$

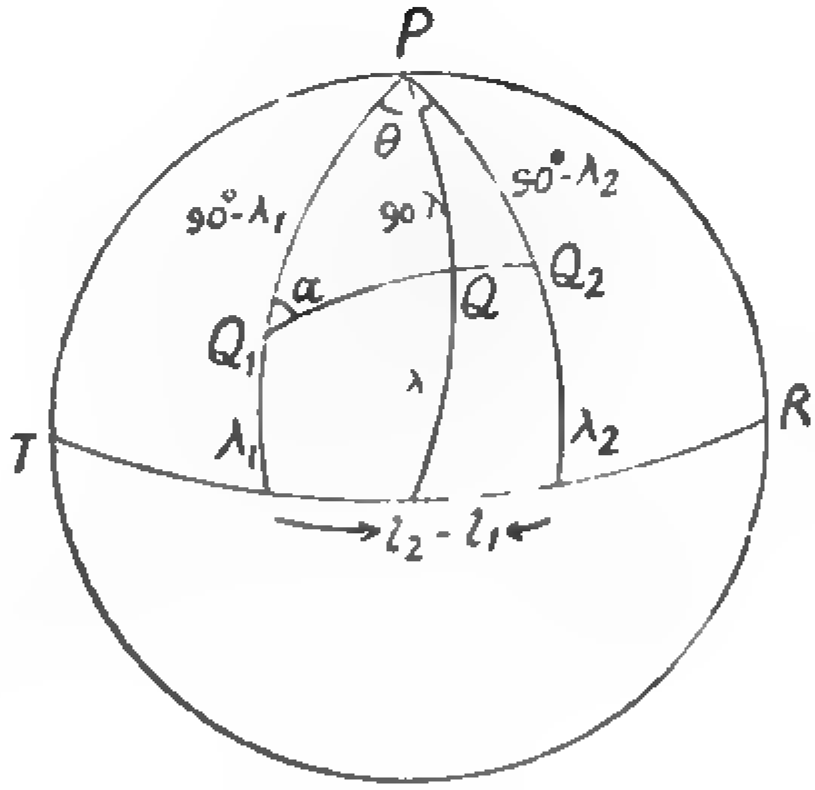


Fig. 74

$\cos \frac{L}{R} = \sin \lambda_1 \sin \lambda_2 \{1 + \tan^2 \phi\}$ [by the given relation].

$\therefore \quad \frac{L}{R} = \cos^{-1} \{\sin \lambda_1 \sin \lambda_2 \sec^2 \phi\}$

or $\quad L = R \cos^{-1} \{\sin \lambda_1 \sin \lambda_2 \sec^2 \phi\}.$

Again suppose that λ be the maximum latitude of the point Q say so that $PQ = 90 - \lambda$. Let $\angle Q_1PQ = \theta$ (variable) and $\angle PQ_1Q = \alpha$ say (constant).

Now from $\triangle PQQ_1$ by applying the formula of consecutive four, we get

$\cos (90 - \lambda_1) \cos \theta = \sin (90 - \lambda_1) \cot (90 - \lambda) - \sin \theta \cot \alpha$

or $\quad \sin \lambda_1 \cos \theta = \cos \lambda_1 \tan \lambda - \sin \theta \cot \alpha. \qquad \ldots(1)$

For Max. value of λ, $\frac{d\lambda}{d\theta}$ should be zero. Hence differentiating (1) and putting $\frac{d\lambda}{d\theta} = 0$, we get

$$-\sin \theta \sin \lambda_1 = -\cos \theta \cot \alpha; \quad \therefore \quad \tan \theta = \frac{\cot \alpha}{\sin \lambda_1}.$$

$$\therefore \quad \sin \theta = \frac{\cot \alpha}{\sqrt{(\cot^2 \alpha + \sin^2 \lambda_1)}},$$

$$\cos \theta = \frac{\sin \lambda_1}{\sqrt{(\cot^2 \alpha + \sin^2 \lambda_1)}}$$

Putting for $\sin\theta$ and $\cos\theta$ in (1), we get

$$\cos\lambda_1 \tan\lambda = \sin\lambda_1 \frac{\sin\lambda_1}{\sqrt{(\cot^2\alpha+\sin^2\lambda_1)}} + \cot\alpha . \frac{\cot\alpha}{\sqrt{(\cot^2\alpha+\sin^2\lambda_1)}}$$

$$=\sqrt{(\sin^2\lambda_1+\cot^2\alpha)}.$$

$$\therefore\quad \tan\lambda = \frac{\sqrt{(\sin^2\lambda_1+\cot^2\alpha)}}{\cos\lambda_1}$$

$$\therefore\quad \cos\lambda = \frac{\cos\lambda_1}{\sqrt{(\cos^2\lambda_1+\sin^2\lambda_1+\cot^2\alpha)}}$$

or $$\cos\lambda = \frac{\cos\lambda_1}{\sqrt{(1+\cot^2\alpha)}} = \cos\lambda_1 \sin\alpha. \qquad \ldots(3)$$

Now we have to eliminate α for which apply sine formula on $\triangle PQ_1Q_2$ and we get

$$\frac{\sin L/R}{\sin(l_2-l_1)} = \frac{\sin(90-\lambda_2)}{\sin\alpha}$$

or $$\sin\alpha = \cos\lambda_2 \sin(l_2-l_1) \operatorname{cosec} L/R.$$

Putting the value of $\sin\alpha$ in (3), we get

$$\lambda = \cos^{-1}\{\cos\lambda_1 \cos\lambda_2 \sin(l_2-l_1) \operatorname{cosec} L/R\}.$$

7. *A port is in latitude l (north) and longitude λ (east). Show that the longitude of places on the equator distant δ from the port are* $\lambda \pm \cos^{-1}(\cos\delta \sec l)$ **(Agra 56 Dacca 1950)**

$GM=\lambda$ given and $PM=l$.

Let A and B be two places on the equator such that $AP=BP=\delta$.

Clearly $AM=MB$. Also from right-angled triangle APM, we have

$$\cos\delta = \cos l . \cos AM,$$

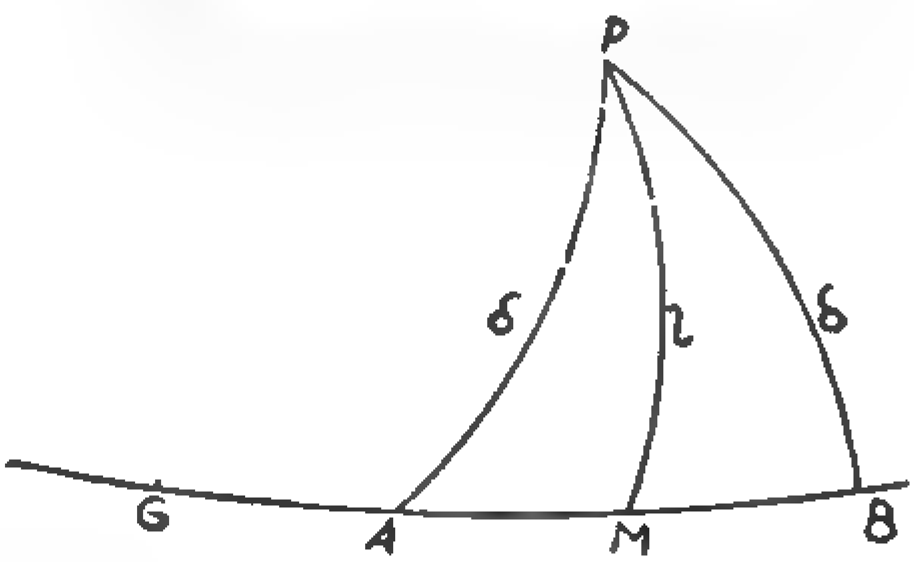


Fig. 75

$\cos AM = \cos\delta . \sec l, \quad \therefore \quad AM = \cos^{-1}(\cos\delta . \sec l)$

or $\quad GM - GA = \cos^{-1}(\cos\delta \sec l).$

Longitude of $A = GA = \lambda - \cos^{-1}(\cos\delta \sec l).$

Longitude of $B = GB = GM + MB = GM + AM$

$= \lambda + \cos^{-1}(\cos\delta \sec l).$

Hence the longitudes are $\lambda \pm \cos^{-1}(\cos\delta \sec l).$

8. *Two places on the earth's surface are distant, one θ from the pole and the other θ from the equator and their difference of longitude is ϕ. Show that the angular distance between them is*

$$\cos^{-1}\left(\sin 2\theta \cos^2 \frac{\phi}{2}\right).$$

$AP = \theta = BD.$

$\therefore \quad BP = 90 - \theta.$

$\cos AB = \cos\theta \cos(90 - \theta)$

$\quad + \sin\theta . \sin(90 - \theta)\cos\phi.$

$\therefore \quad \cos AB = \sin\theta \cos\theta$

$\quad \times (1 + \cos\phi)$

$= \sin\theta \cos\theta \, (2\cos^2 \phi/2)$

$= \sin 2\theta . \cos^2 \phi/2.$

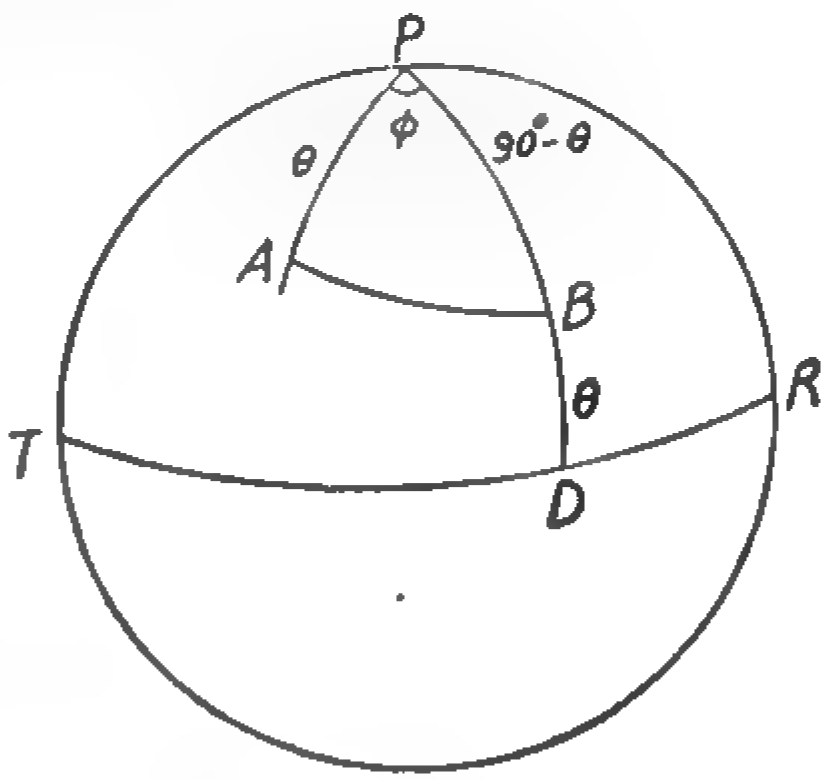


Fig. 76

$\therefore$ Angular distance between A and B is

$AB = \cos^{-1}(\sin 2\theta \cos^2 \phi/2).$

9. *A and B are two places in the northern hemisphere, whose latitudes are λ and λ' and the difference of their longitudes l (where l is supposed to be less than 90°). Show that if a ship sailing by the shortest course from A so B increases her latitude the whole way, $\tan\lambda \cot\lambda'$ must not be greater than $\cos l$.*

[Utkal 54, Nagpur 58, Agra 52]

Join A and B by a great circle arc and S be the position of the ship where its latitude is ϕ, say and suppose that angle $APS=\theta$. If $\angle PAS=\alpha$, then applying the formula of consecutive four, we get

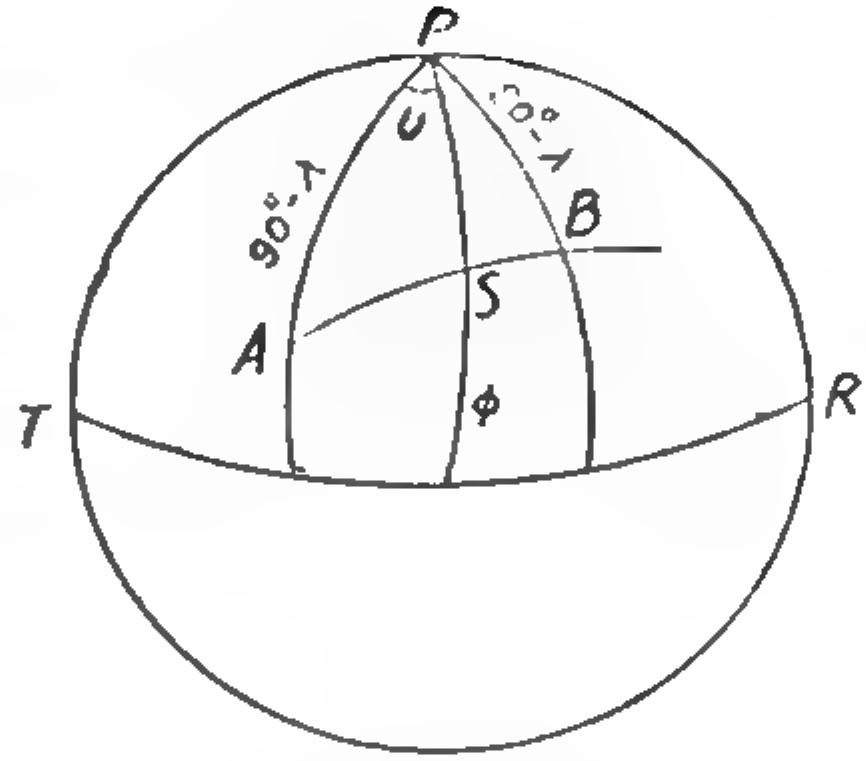


Fig. 77

$$\cos(90-\lambda)\cos\theta=\sin(90-\lambda)\cot(90-\phi)-\sin\theta\cot\alpha.$$

$$\therefore\quad \sin\lambda\cos\theta=\cos\lambda\tan\phi-\sin\theta\cot\alpha. \qquad \ldots(1)$$

Differentiating w. r. t. θ, we get

$$-\sin\lambda\sin\theta=\cos\lambda\sec^2\phi\,\frac{d\phi}{d\theta}-\cos\theta\cot\alpha$$

or
$$\frac{d\phi}{d\theta}=\frac{\cos\theta\cot\alpha-\sin\lambda\sin\theta}{\cos\lambda\sec^2\phi} \qquad \ldots(2)$$

Now $\frac{d\phi}{d\theta}$ stands for rate of change of ϕ w. r. t. θ. If ϕ increases the whole way from A to B, then for values of θ lying between 0 and l, $\frac{d\phi}{a0}$ should not be $-$ive which means that $\frac{d\phi}{d\theta}$ should be greater than or equal to zero.

Hence from (2), we get

$$\cos\theta\cot\alpha-\sin\lambda\sin\theta\geqslant 0.$$

Now putting for $\cot\alpha$ from (1), we get

$$\cos\theta\left\{\frac{\cos\lambda\tan\phi-\sin\lambda\cos\theta}{\sin\theta}\right\}\geqslant\sin\lambda\sin\theta$$

or $\cos\theta\cos\lambda\tan\phi\geqslant\sin\lambda(\cos^2\theta+\sin^2\theta)$ or $\geqslant\sin\lambda$

or
$$\cos\theta\geqslant\tan\lambda\cot\phi. \qquad \ldots(3)$$

Now θ increases as the ship moves from A to B therefore $\cos\theta$ decreases.

Hence the above condition should be satisfied when S is at B *i e.* when $\phi=\lambda'$ and $\theta=l$.

$$\therefore \quad \cos l \geqslant \tan \lambda \cot \lambda'.$$

In other words the above condition amounts to that $\tan \lambda \cot \lambda'$ must not be greater than $\cos l$.

10. *A ship starts from a point on the equator and sails in a great circle cutting the equator at an angle of 45°; find how much she has changed her longitude when she has reached a latitude* $\tan^{-1} \frac{1}{2}$. **(Utkal 56, 59, Gorakhpur 59, Sagar 52)**

Let l be the latitude of the ship when the ship is at B so that $l=\tan^{-1} \frac{1}{2}$ or $\tan l=\frac{1}{2}$. Let the longitude change by λ so that arc $AD=\lambda$. From right-angled triangle ABD, we get

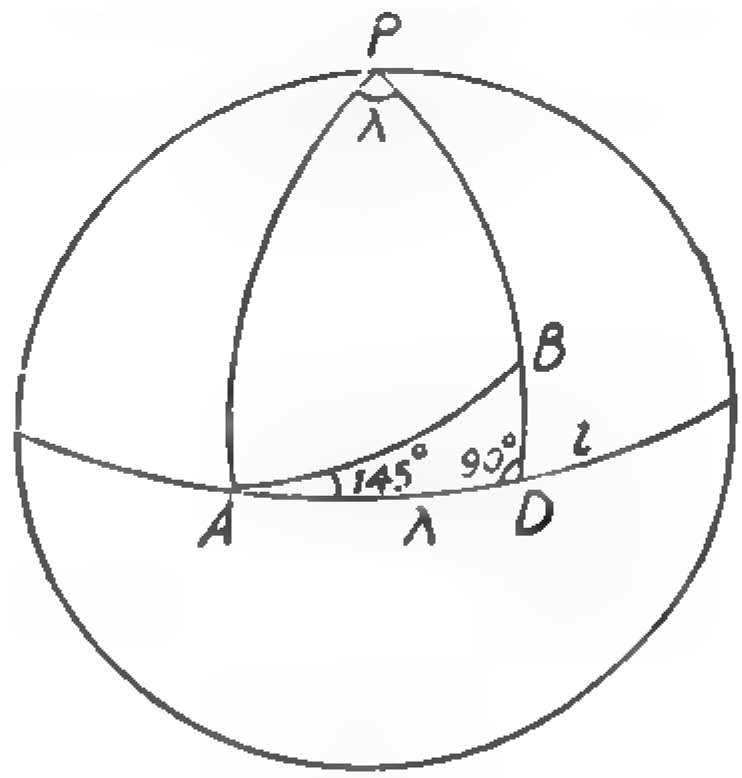


Fig. 78

$$\sin AD=\tan l.\cot BAD$$
$$=\tfrac{1}{2}.\cot 45°=\tfrac{1}{2}.$$
$$\therefore \quad AD=30°.$$

UTKAL UNIVERSITY B. Sc. (Hons.) PAPERS

1954

1. (a) If the three sides of a spherical triangle be halved and a new triangle be formed, prove that the angle θ between the new sides $\frac{1}{2}b$ and $\frac{1}{2}c$ is given by

$$\cos\theta = \cos A + \tfrac{1}{2}\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c\sin^2\theta.$$

(b) If θ, ϕ, ψ denote the distances of the corners A, B, C respectively from the point of intersection of arcs bisecting the angles of the spherical triangle ABC, show that

$$\cos\theta\sin(b-c) + \cos\phi\sin(c-a) + \cos\psi\sin(a-b) = 0.$$

2. (a) Show how to solve a spherical triangle right-angled at C having given a side a and the opposite angle A.

(b) A and B are two places in the Northern hemisphere, whose latitudes are λ and λ', and the difference of their longitudes is l ($<90°$); show that if a ship sailing by the shortest course from A to B increases her latitude on the whole way, $\tan\lambda\cot\lambda'$ must not be greater than $\cos l$.

3. (a) If c_1, c_2 be the two values of the third side when A, a, b are given and the triangle ABC is ambiguous, show that

$$\tan\frac{c_1}{2}\tan\frac{c_2}{2} = \tan\frac{b-a}{2}\tan\frac{b+a}{2}.$$

(b) In a spherical triangle if $A = B = 2C$, show that

$$\cos a\cos\tfrac{1}{2}a = \cos(c + \tfrac{1}{2}a).$$

4. (a) In a spherical triangle ABC, prove that

(i) $$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

(ii) $$\cos\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\tfrac{1}{2}C.$$

(b) If $A=a$, show that

$$\tan \tfrac{1}{2}a=\frac{\tan \frac{1}{2}b \sim \tan \frac{1}{2}c}{1-\tan \frac{1}{2}b \tan \frac{1}{2}c}.$$

1955

1. (a) In a spherical triangle, show that

$\sin b \sin c+\cos b \cos c \cos A=\sin B \sin C-\cos B \cos C \cos a.$

(b) In a spherical triangle if arcs be drawn from the vertices to the middle points of the opposite sides, and if λ, λ' be the two parts of the one which bisects the side a,

show that $$\frac{\sin \lambda}{\sin \lambda'}=2 \cos \frac{a}{2}.$$

2. (a) State Napier's Rules, and deduce from them that in a spherical triangle ABC, in which

$A=90°$, $\cos a=\cos b \cos c$ and $\cot B=\cot b \sin c$.

(b) In a spherical triangle if

$$A=\frac{\pi}{5},\ B=\frac{\pi}{3},\ C=\frac{\pi}{2},$$

show that $$a+b+c=\frac{\pi}{2}.$$

3. (a) Show how to solve a spherical triangle having given the perimeter and the sum of two angles and the third angle.

(b) In a spherical triangle, if $A=B=2C$, show that

$$8 \sin\left(a+\frac{c}{2}\right) \sin^3 \frac{c}{2} \cos \frac{c}{2}=\sin^3 a$$

4. (a) The arc of a great circle bisecting the sides AB, AC of a spherical triangle cuts BC produced at Q; show that

$$\cos AQ \sin \frac{a}{2}=\sin \frac{c-b}{2} \sin \frac{c+b}{2}.$$

(b) Two ports are in the same latitude l, their difference of longitude being 2λ. Show that the distance saved in sailing from one port to the other along a great circle,

instead of due west or east is $2r\{\lambda \cos l - \sin^{-1}(\sin\lambda \cos l)\}$, where r is the radius of the earth.

1956

1. (a) Prove that in any spherical triangle ABC,

$$\cot b \sin c = \cot B \sin A + \cos c \cos A. \quad (\S\ 8\ \text{P. } 25)$$

(b) In a spherical triangle ABC, if $\theta, \phi\ \psi$ be the arcs bisecting the angles A, B, C respectively and terminated by the opposite sides, show that

$$\cot\theta . \cos\frac{A}{2} + \cot\phi\cos\frac{B}{2} + \cot\psi\cos\frac{C}{2}$$
$$= \cot a + \cot b + \cot c.$$

2. (a) Prove that in a spherical triangle ABC,

$$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2}.$$

(b) In a right-angled spherical triangle ABC, if δ be the length of the arc drawn from C perpendicular to the hypotenuse AB, show that

$$\cot\delta = \sqrt{(\cot^2 a + \cot^2 b)}.$$

3. If c_1, c_2 be the two values of the third side when A, a, b are given and the triangle ABC is ambiguous, show that

$$\tan\frac{c_1}{2}\tan\frac{c_2}{2} = \tan\frac{b-a}{2}\tan\frac{b+a}{2}.$$

4. (a) If the sum of the two angles of a spherical triangle is less than π, show that the sum of the opposite sides is less than the semi-circumference of a great circle.

(b) A ship starts from a point on the equator and sails in a great circle, cutting the equator at an angle of 45°; find how much she has changed her longitude when she has reached a latitude $\tan^{-1}(\frac{1}{2})$.

1957

1. (a) Express the sine of an angle of a spherical triangle in terms of the trigonometrical functions of the sides.

(b) If throhgh any point P on a sphere three great circles are drawn, cutting the sides of a triangle at angles x, y, z ; $x_1, y_1\ z_1$; $x_2, y_2\ z_2$ respectively, prove that

$$\begin{vmatrix} \cos x & \cos y & \cos z \\ \cos x_1 & \cos y_1 & \cos z_1 \\ \cos x_2 & \cos y_2 & \cos z_2 \end{vmatrix} = 0.$$

2. (a) Prove that in a spherical triangle, if $c=90°$,

$$\tan a . \tan b + \sec C = 0.$$

(c) If the side c of a spherical triangle be a quadrant, prove that $\sin^2 p = \cot\theta \cot\phi$ where p is the perpendicular on c, and θ and ϕ are the segments of the vertical angle.

3. (a) Prove that the sides of a right-angled spherical triangle must be each less than a quadrant or two of them must be each greater than a quadrant.

(b) If ϕ is the angle between the bisector of the vertical angle C and the perpendicular from it to the opposite side, prove that $\tan\phi = \dfrac{\sin(a-b)}{\sin(a+b)}$.

4. If $A=36°$, $B=60°$, show that

(i) $a+b+c=90°$.

(ii) $\cot a - \cot b = 1$.

1959

1. (a) Prove that, in any spherical triangle,

$$\cot a \sin b = \cot A \sin C + \cos b \cos C. \quad (\S\ 8\ \text{P. } 25)$$

(b) In a spherical triangle, if θ, ϕ, ψ be the arcs bisecting the angles A, B, C respectively, and terminated by opposite sides, show that

$$\cot\theta \cos(A/2) + \cot\phi \cos(B/2) + \cot\psi \cot(C/2)$$
$$= \cot a + \cot b + \cot c.$$

2. (a) In a spherical triangle, if arcs be drawn from the angles to the middle points of the opposite sides, and if

α, α' be the two parts of the one which bisects the side a, shew that
$$\frac{\sin \alpha}{\sin \alpha'}=2 \cos \frac{a}{2}.$$

(b) If a, b, c are known, c being a quadrant, determine the angles; show that if δ be the perpendicular on c from the opposite angles, $\cos^2 \delta=\cos^2 a+\cos^2 b$.

3. (a) In a spherical triangle, if
$$A=\pi/5,\ B=\pi/3,\ C=\pi/2,$$
shew that
$$a+b+c=\pi/2.$$

(b) A ship starts from a point on the equator and sails in a great circle, cutting the equator at an angle of 45°; find how much she has changed her longitude when she has reached a latitude of $\tan^{-1}(\frac{1}{2})$.

4. (a) Solve the spherical triangle having given two sides and the angle opposite to one of them.

(b) If c_1, c_2 be two values of the third side when A, a, b are given and the triangle is ambiguous, shew that
$$\tan\frac{c_1}{2}\tan\frac{c_2}{2}=\tan\frac{b-a}{2}\tan\frac{b+a}{2}.$$

VIKRAM UNIVERSITY M. A. & M. Sc. PAPERS

1959

1. (a) In a spherical triangle ABC, prove that

(i) $\tan \frac{1}{2}(A+B) \tan \frac{1}{2}C$

$$=\cos \tfrac{1}{2}(a-b).\sec \tfrac{1}{2}(a+b).$$

(ii) $\sin c \cos B = \sin a \cos b - \cos a \sin b \cos C.$

(§ 6. P. 23)

(b) If corresponding angles of a triangle ABC and its polar triangle are equal, show that

$$\sec^2 A + \sec^2 B + \sec^2 C + 2 \sec A \sec B \sec C = 1.$$

2. (a) Ox, Oy are two great circles of a sphere at right angles to each other, P is any point in AB another great circle. OC which is p is the arc perpendicular to AB from O, making the angle COx which is α with Ox. PM and PN are arcs perpendicular to Ox and Oy respectively. Show that if $OM=x$ and $ON=y$, then

$$\cos \alpha . \tan x + \sin \alpha \tan y = \tan p.$$

(b) If δ be the length of the arc drawn from C perpendicular to AB in any triangle, prove

$$\cos \delta = \text{cosec } c \, [\cos^2 a + \cos^2 b - 2 \cos a \cos b \cos c]^{1/2}.$$

NAGPUR UNIVERSITY B. Sc. PAPERS

1956

1. Define Polar triangles Show that if one triangle be the polar triangle of another, the latter will be the polar triangle of the former and that the sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle.

Hence or otherwise prove that in a spherical triangle ABC,

$$\cos A+\cos B.\cos C=\sin B.\sin C \cos a.$$

2. (a) Prove Delambre's analogy :

$$\frac{\cos \frac{A-B}{2}}{\sin \frac{C}{2}}=\frac{\sin \frac{a+b}{2}}{\sin \frac{c}{2}}.$$

(b) In a spherical triangle, if θ, ϕ, ψ be the arcs bisecting the angles A, B, C respectively and terminated by the opposite sides, show that

$$\cot \theta \cos \frac{A}{2}+\cot \phi \cos \frac{B}{2}+\cot \psi \cos \frac{C}{2}$$
$$=\cot a+\cot b+\cot c.$$

3. (a) ABC is a great circle of a sphere; AA', BB', CC' are arcs of great circles drawn at right angles to ABC and reckoned positive when they lie on the same side of it. Show that the condition that A', B', C' should lie in a great circle is

$$\tan AA' \sin BC+\tan BB' \sin CA+\tan CC' \sin AB=0.$$

(b) If D be any point in the side BC of a spherical triangle ABC, show that

$$\cos AD.\sin BC=\cos AB \sin DC+\cos AC \sin BD.$$

4. Discuss completely the ambiguity in the solution of a right-angled spherical triangle ABC, given side a and the angle A with $C=90°$.

Solve the triangle if

$$a=42°\ 18'\ 45''$$
$$A=46°\ 15'\ 25''$$
$$C=90°.$$

1957

1. Compare the arc of a small circle subtending any angle at the centre of the circle with the arc of a great circle subtending the same angle at its centre.

Two places on the same parallel of latitude differ in longitude by 30°. If their common latitude be 60°, find the distance between them due east or west, the earth's radius being 3960 miles.

2. In a spherical triangle ABC, show that

$$\cot a \sin b=\cot A \sin C+\cos b \cos C. \quad (\S\ 8.\ P.\ 25)$$

If δ_1, δ_2 and δ_3 denote the bisectors of the external angles of a spherical triangle, show that

$$\cot \delta_1 \sin \tfrac{1}{2}A+\cot \delta_2 \sin \tfrac{1}{2}B+\cot \delta_3 \sin \tfrac{1}{2}C=0.$$

3. Solve any three of the following identities :

(a) $\dfrac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}=\dfrac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c}.$

(b) $\tan \frac{1}{2}(A-a) \tan \frac{1}{2}(B+b)=\tan \frac{1}{2}(B-b) \tan \frac{1}{2}(A+a).$

(c) If $C=\dfrac{\pi}{2}$, then

$$2 \sin^2 \tfrac{1}{2}c=\sin^2 \tfrac{1}{2}(a+b)+\sin^2 \tfrac{1}{2}(a-b).$$

(d) If $C=A+B$, then

$$1-\cos a-\cos b+\cos c=0.$$

4. Taking two great circles OX and OY at right angles to each other as axes of reference, find the equation of any

great circle in terms of the perpendicular to it from O and the angle made by this perpendicular with OX.

If DE be an arc of a great circle bisecting the sides AB, AC of a spherical triangle at D and E, P the pole of DE, and PB, PD, PE, PC be joined by arcs of great circles, shew that
$$\angle BPC = 2\angle DPE.$$

1958

1. (a) In a spherical triangle ABC show that
$$\cos \tfrac{1}{2}A = \sqrt{\left\{\frac{\sin s \sin (s-a)}{\sin b \sin c}\right\}},$$
where $s = \frac{1}{2}(a+b+c)$.

(b) AB, CD are quadrants on the surface of a sphere intersecting at E, the extremities being joined by great circles; show that
$$\cos AEC = \cos AC \cos BD - \cos BC \cos AD.$$

2. (a) In a spherical triangle ABC in which $C = \frac{1}{2}\pi$, show that
$$\tan \tfrac{1}{2}A \sin a = \sin c - \cos a \sin b.$$

(b) In a spherical triangle ABC, if θ, ϕ, ψ be the arcs of great circles drawn from A, B, C perpendicular to the opposite sides, show that
$$\sin a \sin \theta = \sin b \sin \phi = \sin c \sin \psi$$
$$= \sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}.$$

3. (a) In a spherical triangle ABC, if C be a right angle and D the middle point of AB, show that
$$4 \cos^2 \tfrac{1}{2}c \sin^2 CD = \sin^2 a + \sin^2 b.$$

(b) Perpendiculars are drawn from the angles A, B, C of a spherical triangle, meeting the opposite sides at D, E, F respectively; show that
$$\tan BD \tan CE \tan AF = \tan DC \tan EA \tan FB.$$

4. (a) If ABC be a spherical triangle, right-angled at C, and $\cos A = \cos^2 a$, show that if A be not a right angle, $b+c = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, according as b and c are both less or both greater than $\frac{1}{2}\pi$.

(b) A and B are two places in the northern hemisphere whose latitudes are λ and λ', and the difference of their longitudes l (where l is supposed to be less than 90°); show that, if a ship sailing by the shortest course from A to B increases her latitude the whole way, $\tan\lambda \cot\lambda'$ must not be greater than $\cos l$.

1961

1. In a spherical triangle ABC, show that

$$\cot a \sin b = \cot A \sin C + \cos b \cos C. \quad (\S\ 8.\ P.\ 25)$$

In a spherical triangle if θ, ϕ, ψ be the arcs bisecting the angles A, B, C respectively and terminated by the opposite sides, show that

$$\cot\theta \cos\frac{A}{2} + \cot\phi \cos\frac{B}{2} + \cot\psi \cos\frac{C}{2}$$
$$= \cot a + \cot b + \cot c$$

2. Prove Delambre's Analogies.

$$\frac{\sin\frac{A+B}{2}}{\cos\frac{C}{2}} = \frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}$$

Show that in a spherical triangle ABC,

$$\frac{\sin(A+B)}{\sin C} = \frac{\cos a + \cos b}{1+\cos c}.$$

3. In a spherical triangle ABC if $C = 90°$, prove that

$$\tan b = \cos A \tan C.$$

In a spherical triangle ABC right angled at C, if CD is the great circle drawn through C perpendicular to the

hypotenuse AB, prove that

$$\sin^2 CD = \tan AD \tan DB.$$

4. Compare the arc of a small circle on a sphere subtending any angle at the centre of the circle with the arc of a great circle subtending the same angle at its centre.

Two ports are in the same parallel of latitude, their common latitude l and their difference of longitude 2λ, show that the saving of the distance in sailing from one to the other on the great circle instead of sailing due east or west is $2r\,[\lambda \cos l - \sin^{-1}(\sin\lambda \cos l)]$,
λ being expressed in circular measure and r being the radius of the earth.

AGRA UNIVERSITY M. A. AND M. Sc. PAPERS

1956

1. (a) Starting from the cosine formlua, establish that in any spherical triangle ABC,

$$\frac{\sin(A+B)}{\sin C}=\frac{\cos a+\cos b}{1+\cos c}.$$

(b) If in a spherical triangle ABC, the medians from A and B are equal, prove that either $a=b$ or

$$\sin^2\frac{c}{2}=\cos^2\frac{a}{2}+\cos\frac{a}{2}\cos\frac{b}{2}+\cos^2\frac{b}{2}.$$

2. (a) State and prove Napier's rules of circular parts for the solution of a right-angled spherical triangle.

(b) If in a triangle $a=b=\frac{1}{3}\pi$ and $c=\frac{1}{2}\pi$, prove that

$$A+B+C=\pi+\cos^{-1}\tfrac{7}{9}.$$

1957

1. (a) In any spherical triangle ABC, prove that, if

$$A+B+C=2S,\ \tan\frac{a}{2}=\sqrt{\left\{-\frac{\cos S.\cos(S-A)}{\cos(S-B)\cos(S-C)}\right\}}.$$

(b) If D is any point on the base BC, prove that

$$\sin A.\cot AD=\cot b.\sin BAD+\cot c.\sin DAC.$$

Hence obtain the length of the external bisector of angle A intercepted between the vertex A and the base BC.

2. (a) If ABC be a spherical triangle right-angled at C and $\cos A=\cos^2 a$, show that if A be not a right angle $b+c=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ according as b and c are both less or both greater than $\frac{1}{2}\pi$.

(b) Perpendiculars are drawn from the angles A, B, C of a triangle meeting the opposite sides at D, E, F respectively. Show that

$$\tan BD.\tan CE.\tan AF=\tan DC.\tan EA.\tan FB.$$

1958

1. (a) In any spherical triangle ABC, prove that
$$\frac{\sin(A+B)}{\sin C}=\frac{\cos a+\cos b}{1+\cos c}\;;\text{ and }\;\frac{\sin(a+b)}{\sin c}=\frac{\cos A-\cos B}{1-\cos C}.$$

(b) Show that in sailing from one meridian to a place in the same latitude on another meridian, the distance saved by sailing along a great circle instead of due east or west is a maximum for latitude $\cos^{-1}\sqrt{(\operatorname{cosec}^2\lambda-1/\lambda^2)}$ where 2λ is the difference of longitude between the meridians

2. (a) A spherical triangle $A'B'C'$ is constructed with sides $a/2, b/2, c/2$ where a, b, c are the sides of another spherical triangle ABC. Show that
$$\cos A=\cos A'-\tfrac{1}{2}\tan b/2\tan c/2\sin^2 A'.$$

(b) If α, β are the arcs drawn from the right angle C of a spherical triangle ABC, respectively perpendicular to and bisecting the hypotenuse c, show that
$$\sin^2 c/2\,(1+\sin^2\alpha)=\sin^2\beta.$$

1959

1. (a) In a spherical triangle; if θ, ϕ, ψ be the arcs bisecting the angles A, B, C respectively and terminated by the opposite sides, show that
$$\cot\theta\cos\frac{A}{2}+\cot\phi\cos\frac{B}{2}+\cot\psi\cos\frac{C}{2}=\cot a+\cot b+\cot c.$$

(b) Prove that $\cos a\tan B+\cos b\tan A+\tan C$
$$=\cos a\cos b\tan A\tan B\tan C.$$

2. (a) In a spherical triangle, if C be a right angle and D the middle point of AB, show that
$$4\cos^2\frac{c}{2}\sin^2 CD=\sin^2 a+\sin^2 b.$$

(b) Perpendiculars are drawn from the angles A, B, C of any spherical triangle, meeting the opposite sides at D, F, E respectively. Show that

$$\tan BD\tan CE\tan AF=\tan DC\tan EA\tan FB.$$

1960

1. (a) Define a *polar triangle* and show that if one triangle be the polar triangle of another, the later will be the polar triangle of the former (triangles under consideration being spherical triangles).

(b) If α and β be the arcs drawn from the right angle respectively perpendicular to and bisecting the hypotenuse c, show that

$$\sin^2 \tfrac{1}{2}c\,(1+\sin^2 \alpha)=\sin^2 \beta.$$

2. (a) In a spherical triangle ABC, if A, B, C denote the angles and a, the side opposite to the angle A, show that

$$\sin a \sin B \sin C$$
$$=2\sqrt{\{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)\}}.$$

where $2S=A+B+C$

(b) If D be any point in the side BC of a spherical triangle, prove that

$$\cos AD \sin BC=\cos AB \sin DC+\cos AC \sin BD.$$

Agra 61

1. Compare the arc of a small circle on a sphere subtending any angle at the centre of the circle with the arc of a great circle subtending the same angle at its centre.

Perpendiculars are drawn from the angle A, B, C of any triangle meeting the opposite sides at D, E, F respectively. Show that

$$\tan BD.\tan CE.\tan AF=\tan DC.\tan EA.\tan FB.$$

2. In a spherical triangle ABC, prove that

$$\tan \frac{A+B}{2}=\cos \frac{a-b}{2} \sec \frac{a+b}{2} \cot \frac{C}{2}.$$

In a spherical triangle ABC if $A=\frac{\pi}{5}$, $B=\frac{\pi}{3}$ and and $C=\frac{\pi}{2}$, show that $a+b+c=\frac{\pi}{2}$.

AGRA UNIVERSITY PAPER
1962

1. Prove that the sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle (the triangles being spherical).

OAA_1 is a spherical triangle right-angled at A_1 and acute angled at A. The arc A_1A_2 of a great circle is drawn perpendicular to OA, then A_2A_3 is drawn perpendicular to OA_1, and so on. Show that A_nA_{n+1} vanishes when n becomes infinite, and find the value of

$$\cos AA_1 . \cos A_1A_2 . \cos A_2A_3 \ldots \text{to infinity.}$$

2. In a spherical triangle ABC prove that

(i) $\sin b \sin c \sin A = 2n$, and

(ii) $\sin B \sin C \sin a = 2N$

where n and N have their usual meanings.

In a spherical triangle ABC, prove that

$$\cos a \tan B + \cos b \tan A + \tan C$$
$$= \cos a \cos b \tan A \tan B \tan C.$$

VIKRAM UNIVERSITY PAPER
1962

1. (a) In a spherical triangle ABC, prove that

(i) $\cot a \sin c = \cot A \sin B + \cos c \cos B$.

(ii) $\tan \dfrac{A-B}{2} = \dfrac{\sin \frac{a-b}{2}}{\sin \frac{a+b}{2}} \cot \dfrac{C}{2}$

(b) If corresponding angles of a triangle ABC and its polar triangle are equal, show that

$$\sec^2 A + \sec^2 B + \sec^2 C + 2 \sec A \sec B \sec C = 1.$$

2. (a) In a spherical triangle show that if θ, ϕ, ψ be the arcs of great circles drawn from A, B, C perpendicular to the opposite sides,

$\sin a \sin \theta = \sin b \sin \phi = \sin c \sin \psi$

$$= [1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c]^{1/2}$$

(b) A ship starts from a point on the equator and sails in a great circle cutting the equator at an angle of 45° find how much she has changed her longitude when she has reached a latitude $\tan^{-1}(\frac{1}{2})$.

DELHI UNIVERSITY PAPERS

1957

1. (a) In any spherical triangle ABC, prove that

$$\cos b \cos C = \sin b \cot a - \sin C \cot A.$$

(b) The most southenly latitude reached by the great circle joining a place A on the equator to a place B in south latitude ϕ is ϕ_1. Prove that the difference of longitude between A and B is $90° + \cos^{-1}(\tan \phi \cot \phi_1)$.

2. (a) Prove that in a spherical triangle ABC,

$$\tan \frac{b-c}{2} = \frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)} \tan \frac{a}{2}.$$

(b) If CD be the arc of a great circle drawn through C perpendicular to the hypotenuse AB of a right-angled spherical triangle ABC, prove that

(i) $\cot^2 CD = \cot^2 a + \cot^2 b$, and

(ii) $\sin^2 CD = \tan AD \tan DB$.

1958

1. (a) In any spherical triangle ABC show that

$$\tan \tfrac{1}{2}A = \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}\right\}}.$$

(b) Prove that in the spherical triangle ABC

$$\tan b = \frac{\tan a \cos C + \tan c \cos A}{1 - \tan a \tan c \cos A \cos C}.$$

2. (a) (i) In a spherical triangle, if C be a right angle and D the middle point of AB, show that

$$4\cos^2 \tfrac{1}{2}c \sin^2 CD = \sin^2 a + \sin^2 b.$$

(ii) In a spherical triangle ABC right-angled at C, show that $\sin (A-B)=(\cos b-\cos a)/(1-\cos b \cos a)$.

(b) ABC is a great circle of a sphere; AA', BB', CC' are arcs of great circles drawn at right angles to ABC and reckoned positive when they lie on the same side of it; show that the condition that A', B', C' should lie on a great circle is $\tan AA' \sin BC+\tan BB' \sin CA+\tan CC' \sin AB=0$.

1959

1. (a) In a spherical triangle ABC, prove that

$$\tan \tfrac{1}{2} (A+B)=\frac{\cos \frac{1}{2} (a-b)}{\cos \frac{1}{2} (a+b)} \cot \tfrac{1}{2} C.$$

(b) In a spherical triangle, if θ, ϕ, ψ be the arcs bisecting the angles A, B, C respectively and terminated by the opposite sides, show that

$$\cot \theta \cos \tfrac{1}{2}A+\cot \phi \cos \tfrac{1}{2}B+\cot \psi \cos \tfrac{1}{2}C$$
$$=\cot a+\cot b+\cot c.$$

2. (a) (i) In a spherical triangle ABC, if CX and CY are the internal and external bisectors of C, which is a right angle, prove that $\cot^2 CX+\cot^2 CY=\cot^2 a+\cot^2 b$.

(ii) For a spherical triangle ABC in which C is $\frac{1}{2}\pi$, show that, if $s=\frac{1}{2} (a+b+c)$,

$$\sin s \sin (s-c)=\sin (s-a) \sin (s-b).$$

(b) The position of a point on a sphere, with reference to two great circles at right angles to each other as axes, is determined by the portions θ, ϕ of these circles cut off by great circles through the point, and through two points on the axes, each $\frac{1}{2}\pi$ from their point of intersection; show that if the three points (θ, ϕ), (θ', ϕ), (θ'', ϕ'') lie on the same great circle, $\tan \phi (\tan \theta'-\tan \theta'')+\tan \phi' (\tan \theta''-\tan \theta)$

$$+\tan \phi'' (\tan \theta-\tan \theta')=0.$$

PUNJAB UNIVERSITY PAPER.

1958

1. Establish the relation between the elements of a spherical triangle and those of its polar triangle.

Express $\tan \frac{a}{2}$ in terms of the trigonometrical ratios of the angles A, B, C of a spherical triangle.

If corresponding angles of a triangle ABC and its polar triangle are equal. prove that

$$\sec^2 A+\sec^2 B+\sec^2 C+2 \sec A \sec B \sec C=1.$$

2. (a) Explain Napier's rules for the solution of a right-angled spherical triangle.

Solve a right-angled spherical triangle having given its two angles.

(b) OAA_1 is a spherical triangle right-angled at A_1 and acute-angled at A; the arc A_1A_2 of a great circle is drawn perpendicular to OA; then A_2A_3 is drawn perpendicular to OA_1, and so on. Show that A_nA_{n+1} vanishes when n tends to infinity, and find the value of $\cos AA_1 \cos A_1A_2 \cos A_2A_3 \ldots$ to infinity.

1959

1. Establish cosine formlae for a spherical triangle.

In a spherical triangle ABC prove that

(i) $\tan \frac{1}{2}(A-B)=\dfrac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{1}{2}C.$

(ii) $\dfrac{\sin c}{\sin C}=\left\{\dfrac{1-\cos a \cos b \cos c}{1+\cos A \cos B \cos C}\right\}^{\frac{1}{2}}.$

2. Obtain formulae to solve a spherical triangle ABC right-angled at C when A and B are given. Find the conditions under which the triangle is possible.

If in the above triangle, D be the middle point of AB, prove that

$$4 \cos^2 \tfrac{1}{2}c \sin^2 CD = \sin^2 a + \sin^2 b.$$

1960

1. (a) Establish the following formula : (cosine of inner side) (cosine of inner angle)=(sine of inner side) (cotangent of other side)—(sine of inner angle) (cotangent of other angle)

(b) If in a spherical quadrilateral $ABCD$, the sides AB, BC be denoted by a, b respectively and the angle ABD be θ, show that

$$\tan \theta = \frac{\cos a \sin b - \sin a (\cos b \cos B + \cot C \sin B)}{-\cot A \sin b + \sin a (\cos b \sin B - \cot C \cos B)}.$$

(c) Prove the following theorem for a spherical triangle :

If one angle of a spherical triangle is greater than the other, the side opposite the greater angle is greater than the side opposite to the smaller angle.

2. (a) In a spherical triangle ABC, $C = \frac{\pi}{2}$, $\cos A = \cos^2 a$, $A \neq \frac{\pi}{2}$; prove that $b + c = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$ stating the conditions under which these values are possible.

(b) In a spherical triangle ABC, $A = \frac{\pi}{5}$, $B = \frac{\pi}{3}$ and $C = \frac{\pi}{2}$; prove that $a + b + c = \frac{\pi}{2}$.

(c) If ABC be a spherical triangle right-angled at C, show that $\sin a \tan \frac{1}{2}A - \sin b \tan \frac{1}{2}B = \sin (a - b)$.